Solutions to Exam 2
MATH 304, Section 6

1. (4 points each) Complete each definition. Make sure that you write a complete meaningful sentences containing all necessary conditions.

   a) $w_1, w_2, \ldots, w_n$ is a basis of a vector space W if ... .
   b) The kernel (null space) of a linear transformation $T : U \longrightarrow W$ is ... .
   c) The span $\operatorname{span}\{u_1, \ldots, u_n\}$ is ... .
   d) Vectors $u_1, \ldots, u_m$ are linearly independent if ....
   e) A function $F : V \longrightarrow W$ between vector spaces is called a linear transformation if ... .

   Solution.

   a) The sequence (list) $w_1, w_2, \ldots, w_n$ is a basis of a vector space W if it is both linearly independent and spans the whole vector space W.

   b) The kernel (null space) $\ker(T)$ of a linear transformation $T : U \longrightarrow W$ is the set of all vectors $u \in U$ such that $T(u) = 0$:
   $$\ker(T) = \{u \in U : T(u) = 0\}.$$

   c) $\operatorname{span}\{u_1, \ldots, u_n\}$ is the set of all vectors which are linear combinations of the vectors $u_1, \ldots, u_n$:
   $$\operatorname{span}\{u_1, \ldots, u_n\} = \{a_1u_1 + a_2u_2 + \ldots + a_nu_n : a_1, a_2 \ldots, a_n \text{ are numbers}\}.$$

   d) Vectors $u_1, \ldots, u_m$ are linearly independent if they are all non-zero and none of them can be expressed as the linear combination of the other vectors. Equivalently, $u_1, \ldots, u_m$ are linearly independent if the only numbers $a_1, \ldots, a_m$ such that $a_1u_1 + a_2u_2 + \ldots + a_mu_m = 0$ are $a_1 = a_2 = \ldots = a_m = 0$.

   e) A function $F : V \longrightarrow W$ between vector spaces is called a linear transformation if $F(v_1 + v_2) = F(v_1) + F(v_2)$ and $F(cv_1) = cF(v_1)$ for any vectors v_1, v_2 in V and any scalar c.

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2. (20 points) Let $v_1 = (1, 0, 1, 0)$, $v_2 = (1, -1, 0, 2)$, $v_3 = (1, -3, -2, 6)$, $v_4 = (1, 1, 1, 1)$, $v_5 = (2, -5, -2, 7)$. Among these vectors find a basis of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5\}$. Express each vector v_i as a linear combination of vectors in this basis. What is the dimension of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5\}$?

   Solution. To solve the problem, we form a matrix A whose columns are the vectors $v_1, \ldots, v_5$. The pivot columns of A form a basis of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5\}$ and the non-pivot columns of the reduced row-echelon form R of A provide coefficients for expressing the non-pivot columns of A as linear combinations of the pivot columns. The dimension of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5\}$ is the number of pivot columns of A.

   We have
   $$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & 1 & -5 \\ 1 & 0 & -2 & 1 & -2 \\ 0 & 2 & 6 & 1 & 7 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
   (R was provided at the end of the exam paper). We see that the first, the second, and the fourth columns are the pivot columns. Thus, v_1, v_2, v_4 is a basis of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5\}$. Furthermore, column two of R tells us that $v_3 = -2v_1 + 3v_2$, and column five of R tells us that $v_5 = -v_1 + 4v_2 - v_4$. The dimension of $\operatorname{span}\{v_1, v_2, v_3, v_4, v_5\}$ is equal to 3.

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3. A linear transformation $L_A : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ has matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 4 & -13 & -17 \\ 4 & 8 & -6 & 25 & 29 \\ -1 & -2 & 0 & 10 & 6 \end{bmatrix}.$$

a) (10 points) Find a basis of the image of L_A.

b) (10 points) Find a basis of the kernel of L_A. Verify that the vectors you found are indeed in the kernel.

Solution. a) The pivot columns of A form a basis of the image of L_A. It was given that the reduced row-echelon form of A is

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that the first, the third, and the fourth columns of A are the pivot columns. Thus the vectors $(1, -3, 4, -1), (-1, 4, -6, 0), (3, -17, 29, 6)$ form a basis of the image of L_A.

b) To find a basis of the kernel $\ker(L_A)$, we consider the homogeneous system of linear equations $Rx = 0$. For every free variable x_i we find the solution with $x_i = 1$ and all other free variables equal to 0. These solutions will form a basis of the kernel. Looking at the matrix R we see that x_2 and x_5 are the free variables. The solution with $x_2 = 1, x_5 = 0$ is $u_1 = (-2, 1, 0, 0, 0)$. The solution with $x_2 = 0, x_5 = 1$ is $u_2 = (-4, 0, -2, -1, 1)$. Thus u_1, u_2 is a basis of $\ker(L_A)$.

To check that u_1, u_2 are indeed in the kernel, means to verify that $L_a(u_1) = 0 = L_A(u_2)$. This is the same as checking that

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 4 & -13 & -17 \\ 4 & 8 & -6 & 25 & 29 \\ -1 & -2 & 0 & 10 & 6 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -1 & 1 & 3 \\ -3 & -6 & 4 & -13 & -17 \\ 4 & 8 & -6 & 25 & 29 \\ -1 & -2 & 0 & 10 & 6 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

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4. (14 points) Let $v_1 = (1, -1, -1, 1)$, $v_2 = (-1, 1, 1, 0)$, $v_3 = (0, 1, 0, 1)$ and $W = \operatorname{span}\{v_1, v_2, v_3\}$. Furthermore, let $u = (1, 0, -1, 3)$ and $w = (1, 1, 1, 1)$.

a) Does u belong to W? If yes, express it as a linear combination of v_1, v_2, v_3. If not, explain why.

b) Does w belong to W? If yes, express it as a linear combination of v_1, v_2, v_3. If not, explain why.

Solution. In order to solve the problem we form a matrix A whose columns are the vectors v_1, v_2, v_3, u, w and find its reduced row-echelon form R (which was given at the end of the text of the exam).

We have

$$A = \left[\begin{array}{ccc:cc} 1 & -1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 & 1 \\ -1 & 1 & 0 & -1 & 1 \\ 1 & 0 & 1 & 3 & 1 \end{array}\right] \quad \text{and} \quad R = \left[\begin{array}{ccc:cc} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

From R we see that v_1, v_2, v_3 are linearly independent and that the first column to the right of the dividing line is in the span of the columns to the left of the dividing line (the corresponding system of equations is consistent). Thus u belongs to W and $u = 2v_1 + v_2 + v_3$. This answers part a).

For b), we look at the second column to the right of the dividing line and we see that this column is not in the span of the columns to the left of the dividing line (the corresponding system of equations is inconsistent). Thus w is not in W.

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5. (10 points) Let $T : \mathbb{R}^{10} \longrightarrow \mathbb{R}^{10}$ be a linear transformation such that the composition $T \circ T$ is the zero map (i.e. maps every vector of $\mathbb{R}^{10}$ to 0).

   a) Show that every vector v in the image of T belongs to the the kernel $\ker(T)$ of T.

   b) Can both the kernel of T and the image of T have dimension bigger than 5? Carefully justify your answer.

   c) Show that the rank of the matrix of T does not exceed 5.

   Solution. a) If v is in the image of T then $v = T(w)$ for some $w \in \mathbb{R}^{10}$. Thus $T(v) = T(T(w)) = (T \circ T)(w) = 0$, which means that v is in the kernel of T. In other words, $\text{image}(T) \subseteq \ker(T)$.

   b) By the rank-nullity theorem we know that

   $$\dim(\ker(T)) + \dim(\text{image}(T)) = 10.$$

   If both the kernel of T and the image of T had dimension bigger than 5, the sum on the left would be bigger than 10, which is not possible.

   c) From b) we know that at least one of $\dim(\ker(T))$ and $\dim(\text{image}(T))$ has dimension at most 5. From a) we know that the image $\dim(\text{image}(T))$ is contained inthe kernel $\dim(\ker(T))$, so $\dim(\ker(T)) \geq \dim(\text{image}(T))$. Thus $\dim(\text{image}(T)) \leq 5$.

6. Answer true or false (2 points each).

   a) There are 6 linearly independent vectors in $\mathbb{R}^5$.
   This is FALSE. In a vector space of dimension n every linearly independent sequence (or set) of vectors has at most n elements. Since $\mathbb{R}^5$ has dimension 5, it does not have more than 5 linearly independent vectors.

   b) There is a linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ such that the kernel of T is equal to the image of T.
   This is FALSE. By the rank-nullity theorem, $\dim(\ker(T)) + \dim(\text{image}(T)) = 5$. Since 5 is odd, $\dim(\ker(T))$ and $\dim(\text{image}(T))$ cannot be equal.

   c) $\text{rank}(A) + \text{rank}(A^T)$ is even for every matrix A.
   This is TRUE. We know that, for every matrix A, $\text{rank}(A) = \text{rank}(A^T)$. Thus $\text{rank}(A) + \text{rank}(A^T) = 2\text{rank}(A)$ is even.

   d) Any 7 vectors which span $\mathbb{R}^7$ are linearly independent.
   This is TRUE. If the vectors were linearly dependent, we could remove one of them and the remaining vectors would still span $\mathbb{R}^7$ (going-down theorem). Thus $\mathbb{R}^7$ would be spanned by six vectors. However, vector space of dimension n can not be spanned by fewer than n vectors.

   e) The dimension of the row space of a matrix A is equal to the number of pivot columns of A.
   This is TRUE. We know that both the dimension of the row space of A and the dimension of the column space of A are equal to the rank of A, which is the number of pivot columns of A.

   f) There exist a 5×3 matrix A and a 3×5 matrix B such that AB has rank 4.
   This is FALSE. We know that the row space of AB is contained in the row space of B which has dimension equal to the rank of B. Since rank of B is at most 3, rank of AB is at most 3 as well.

g) There is a 6×8 matrix M which has 4 pivot columns and such that every 4 rows of M are linearly dependent.
This is FALSE. Since very 4 rows of M are linearly dependent, among the rows of M there are at most 3 rows which are linearly independent. This means that the row space of M has dimension at most 3. Since the dimension of the row space of M is equal to the rank of M, the rank of M is at most 3. Since the rank is the number of pivot columns, M can not have 4 pivot columns.

h) The collection of all non-pivot columns of a matrix is always linearly dependent.
This is FALSE. Look at the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The last 2 columns of A are non-pivot, but they are clearly linearly independent.

The following problem is optional. You can earn 10 extra points if you solve it, but work on it only if you are done with all the other problems

7. Let $T : V \longrightarrow V$ be a linear transformation. Suppose v is a vector in V such that $w = T(v) \neq 0$ but $T(w) = 0$. Show that v and w are linearly independent.

Solution. Suppose that v, w were linearly dependent. Since $w \neq 0$, we have $v = cw$ for some scalar c. It follows that $w = T(v) = T(cw) = cT(w) = 0$, a contradiction.

Second method. Suppose that $av + bw = 0$. Then $T(av + bw) = 0$, i.e. $aT(v) + bT(w) = 0$, which is the same as $aw = 0$ (recall that $T(v) = w$ and $T(w) = 0$). Since $w \neq 0$, we have $a = 0$. It follows that $bw = 0$, so $b = 0$. Thus the only a, b such that $av + bw = 0$ are $a = b = 0$. This means that v, w are linearly independent.