# Solutions to Exam I

**Problem 1.** a) The length of the vector $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ equals

$$|\mathbf{v}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}.$$

b) The vertices are $A(0,0)$, $B(0, 3+\sqrt{3})$, $C(3, \sqrt{3})$. Thus $\overrightarrow{AB} =< 0, 3+\sqrt{3} >$, $\overrightarrow{AC} =< 3, \sqrt{3} >$. It follows that $|\overrightarrow{AB}| = \sqrt{0^2 + (3+\sqrt{3})^2} = 3 + \sqrt{3}$, $|\overrightarrow{AC}| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12}$ and

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(3+\sqrt{3})\sqrt{3}}{(3+\sqrt{3})\sqrt{12}} = 1/2,$$

i.e. $\measuredangle A = \pi/3$.

Similarly, $\overrightarrow{BA} =< 0, -3-\sqrt{3} >$, $\overrightarrow{BC} =< 3, -3 >$. It follows that $|\overrightarrow{BA}| = \sqrt{0^2 + (-3-\sqrt{3})^2} = 3 + \sqrt{3}$, $|\overrightarrow{BC}| = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$ and

$$\cos B = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(-3-\sqrt{3})(-3)}{(3+\sqrt{3})3\sqrt{2}} = 1/\sqrt{2} = \sqrt{2}/2,$$

i.e. $\measuredangle B = \pi/4$.

Since the sum of all three angles in a triangle equals $\pi$, we get $\measuredangle C = \pi - \pi/3 - \pi/4 = 5\pi/12$.

c) Recall that vectors $< a, b >$ and $< b, -a >$ are always orthogonal. Thus $< 4, -3 >$ is orthogonal to $< 3, 4 >$. Since $| < 4, -3 > | = 5$, the vector $< 4/5, -3/5 >$ is a unit vector orthogonal to $< 3, 4 >$.

d) Let $\mathbf{a} =< 1, 0, 1 >$ and $\mathbf{b} =< 1, 1, 0 >$. We are looking for vectors $\mathbf{u}$, $\mathbf{w}$ such that $\mathbf{u}$ is parallel to $\mathbf{a}$, $\mathbf{w}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b} = \mathbf{u} + \mathbf{w}$. Since $\mathbf{u}$ is parallel to

$\mathbf{a}$, we may write $\mathbf{u} = t\mathbf{a}$ for some scalar $t$. Taking the dot product of both sides of the equality $\mathbf{b} = t\mathbf{a} + \mathbf{w}$ with $\mathbf{a}$ we get

$$\mathbf{b} \cdot \mathbf{a} = t\mathbf{a} \cdot \mathbf{a} + \mathbf{w} \cdot \mathbf{a} = t\mathbf{a} \cdot \mathbf{a}$$

(since $\mathbf{w} \cdot \mathbf{a} = 0$). Thus $t = \mathbf{b} \cdot \mathbf{a} / \mathbf{a} \cdot \mathbf{a} = 1/2$, i.e. $\mathbf{u} = \mathbf{a}/2 =< 1/2, 0, 1/2 >$. Finally, $\mathbf{w} = \mathbf{b} - \mathbf{u} =< 1/2, 1, -1/2 >$.

**Problem 2.** a) $(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) =< 2, 1, -1 > \times < 1, -2, 3 >=$ $< 1 \cdot 3 - (-1)(-2), -(2 \cdot 3 - (-1) \cdot 1, 2 \cdot (-2) - 1 \cdot 1 >=< 1, -7, -5 >$. Alternatively, you could use the fact that the cross product is distributive with respect to addition.

b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 1 + (-1) = 1.$

c) The volume of the parallelepiped determined by vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ equals $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. For $\mathbf{u} =< 0, 0, 1 >$, $\mathbf{v} =< 1, 0, 1 >$, $\mathbf{w} =< 1, 1, 1 >$ the volume equals 1 .

d) The area of a triangle with vertices $A$, $B$, $C$ equals half of the area of the parallelogram determined by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so it is equal to $|\overrightarrow{AB} \times \overrightarrow{AC}|/2$. In the problem, $A(0, 0, 0)$, $B(1, 0, 1)$, $C(1, 1, 1)$, so the area equals

$$| < 1, 0, 1 > \times < 1, 1, 1 > |/2 = \sqrt{2}/2.$$

**Problem 3.** a) $x^2 + y^2 + z^2 = x - y + z$ may be written as

$$(x - 1/2)^2 + (y + 1/2)^2 + (z - 1/2)^2 = 3/4$$

so this equation describes the sphere with center $(1/2, -1/2, 1/2)$ and radius $\sqrt{3}/2$.

b) Adding the equations $2x - y - z = 0$ and $x - 2y + z = 0$ yields $3x - 3y = 0$, i.e. $x = y$. Thus $z = 2x - y = 2x - x = x$, i.e. $x = y = z$. It follows that the parametric equation is $x = t, y = t, z = t$ and the symmetric equation is $x = y = z$.

c) The plane containing points $A, B, C$ is orthogonal to a vector orthogonal to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, i.e. it is orthogonal to $\overrightarrow{AB} \times \overrightarrow{AC}$. In the problem, $A(1, 0, 1)$, $B(0, 1, 1)$, $C(1, 1, 0)$. Thus $\overrightarrow{AB} \times \overrightarrow{AC} =< -1, 1, 0 > \times < 0, 1, -1 >=< -1, -1, -1 >$. Thus we want an equation of the plane passing through $(1, 0, 1)$ and orthogonal to

$< -1, -1, -1 >$, which is $-(x-1) - y - (z-1) = 0$, i.e. $x + y + z = 2$.

**Problem 4.** a) The curvature of a curve $\mathbf{r}(t)$ is defined as $k(t) = |\mathbf{T}'(t)|/v(t)|$, where $\mathbf{T}(t) = \mathbf{v}(t)/v(t)$ is the unit tangent vector, $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity and $v(t) = |\mathbf{v}'(t)|$ is the speed. It turns out that the curvature of a curve does not depend on the parametrization. This means that the if $\mathbf{r}_1(t)$ is another parametrization of the same curve then at the point $\mathbf{r}(t) = \mathbf{r}_1(t_1)$ the curvature $k(t)$ computed for the first parametrization is the same as the curvature $k(t_1)$ computed for the second parametrization. Thus curvature is a geometric notion which describes how the curve curves at a given point. For plane curves, the inverse of the curvature is equal to the radius of a circle which "fits" best the curve at a given point. The curvature can be computed from the formula $k(t) = |\mathbf{v}(t) \times \mathbf{a}(t)|/v^3(t)$.

b) We have $\mathbf{r}(t) =< 2t - \sin 2t, -\cos 2t, 4\sin t >$.
The velocity $\mathbf{v}(t) = \mathbf{r}'(t) =< 2 - 2\cos 2t, 2\sin 2t, 4\cos t >$.
The speed $v(t) = |\mathbf{v}(t)| = \sqrt{(2-2\cos 2t)^2 + (2\sin 2t)^2 + (4\cos t)^2} =$

$$\sqrt{4 - 8\cos 2t + 4\cos^2 2t + 4\sin^2 2t + 16\cos^2 t} =$$

$$\sqrt{4 + 4(\cos^2 2t + \sin^2 2t) - 8(2\cos^2 t - 1) + 16\cos^2 t} = \sqrt{4+4+8} = 4.$$

The acceleration $\mathbf{a}(t) = \mathbf{v}'(t) =< 4\sin 2t, 4\cos 2t, -4\sin t >$.
The unit tangent vector $\mathbf{T}(t) = \mathbf{v}(t)/v(t) =< (1 - cos 2t)/2, \sin 2t/2, \cos t >$.
The unit normal vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{< \sin 2t, \cos 2t, -\sin t >}{\sqrt{\sin^2 2t + \cos^2 2t + \sin^2 t}} =< \frac{\sin 2t}{\sqrt{1+\sin^2 t}}, \frac{\cos 2t}{\sqrt{1+\sin^2 t}}, \frac{-\sin t}{\sqrt{1+\sin^2 t}} > .$$

The curvature $k(t) = |\mathbf{T}'(t)|/v(t) = \sqrt{1+\sin^2 t}/4$.

c) The velocity of the parametric curve $\mathbf{r}(t) =< t\sin t, t\cos t, \frac{2\sqrt{2}}{3}t^{3/2} >$ equals

$$\mathbf{v}(t) =< \sin t + t\cos t, \cos t - t\sin t, \sqrt{2t} > .$$

The speed $v(t) = \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + (\sqrt{2t})^2} =$

$$\sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + 2t} = \sqrt{1 + t^2 + 2t} = 1+t.$$

The length of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(t)$ is

$$s(t) = \int_0^t v(u)du = \int_0^t (1+u)du = t + t^2/2.$$

From the equation $s = t + t^2/2$ we get $1 + 2s = 1 + 2t + t^2 = (1+t)^2$, hence $t = \sqrt{2s+1} - 1$. The arc-length parametrization is then

$$\mathbf{r}(s) =< (\sqrt{2s+1}-1)\sin(\sqrt{2s+1}-1), (\sqrt{2s+1}-1)\cos(\sqrt{2s+1}-1), \frac{2\sqrt{2}}{3}(\sqrt{2s+1}-1)^{3/2} > .$$

d) A particle moves in the space with acceleration $\mathbf{a}(t) =< 2, 6t, 12t^2 >$. The velocity of the particle equals

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt =< 2t + c_1, 3t^2 + c_2, 4t^3 + c_3 >$$

for some constants $c_1, c_2, c_3$. The condition $\mathbf{v}(1) =< 3, 4, 5 >$ implies that $c_1 = c_2 = c_3 = 1$, i.e. $\mathbf{v}(t) =< 2t+1, 3t^2+1, 4t^3+1 >$. The position of the particle equals

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt =< t^2 + t + d_1, t^3 + t + d_2, t^4 + t + d_3 > .$$

The condition $\mathbf{r}(1) =< 3, 2, 2 >$ implies that $d_1 = 1$, $d_2 = d_3 = 0$, i.e. $\mathbf{r}(t) =< t^2 + t + 1, t^3 + t, t^4 + t >$. Thus, at $t = 0$ the particle is at the point $(1, 0, 0)$.

**Problem 5.** a) The cylindrical coordinates of the point $(1, 1, \sqrt{6})$ are $(r, \theta, z)$, where $r^2 = 1^2 + 1^2 = 2$, $\tan\theta = 1/1 = 1$, $z = \sqrt{6}$. Thus $\theta = \pi/4$ and the cylindrical coordinates are $(\sqrt{2}, \pi/4, \sqrt{6})$.

The spherical coordinates of this point are $(\rho, \phi, \theta)$, where $\theta = \pi/4$ is the same as for the cylindrical coordinates, $\rho^2 = 1^2 + 1^2 + \sqrt{6}^2 = 8$ and $\cos\phi = \sqrt{6}/\rho = \sqrt{6}/\sqrt{8} = \sqrt{3}/2$. Thus $\phi = \pi/6$ and the spherical coordinates are $(2\sqrt{2}, \pi/6, \pi/4)$.

b) The point whose cylindrical coordinates are $(1, \pi/6, 1)$ has Cartesian coordinates $(\cos\pi/6, \sin\pi/6, 1)$. The spherical coordinates are $(\rho, \phi, \theta)$, where $\theta = \pi/6$ (same as for cylindrical coordinates), $\rho^2 = \cos^2\pi/6 + \sin^2\pi/6 + 1 = 2$ and $\cos\phi = 1/\rho = 1/\sqrt{2} = \sqrt{2}/2$. Thus $\phi = \pi/4$ and the spherical coordinates are $(\sqrt{2}, \pi/4, \pi/6)$.

c) A plane curve in polar coordinates has equation $r = \cos\theta$. Since $x = r\cos\theta = \cos^2\theta$, $y = r\sin\theta = \cos\theta\sin\theta$, the curve has parametric equation $\mathbf{r}(\theta) =< \cos^2\theta, \sin\theta\cos\theta >$ in Cartesian coordinates. The velocity is $\mathbf{v}(\theta) =< -2\sin\theta\cos\theta, \cos^2\theta - \sin^2\theta >$. The speed $v(\theta) = \sqrt{(-2\sin\theta\cos\theta)^2 + (\cos^2\theta - \sin^2\theta)^2} = \sqrt{(\cos^2\theta + \sin^2\theta)^2} = 1$. The acceleration $\mathbf{a}(\theta) =< -2(\cos^2\theta - \sin^2\theta), -4\sin\theta\cos\theta >$. Since the parametrization $\mathbf{r}(\theta)$ is a natural (arc-length) parametrization, we have $\mathbf{T}(\theta) = \mathbf{v}(\theta)$ and

$k(\theta) = |\mathbf{T}'(\theta)| = |\mathbf{a}(\theta)|$. Thus

$$k(\theta) = \sqrt{[(-2)(\cos^2\theta - \sin^2\theta)]^2 + (-4\sin\theta\cos\theta)^2} = \sqrt{4(\cos^2\theta + \sin^2\theta)^2} = 2.$$

**Remark.** The computation simplify significantly when the formulas $\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x$ are used.

**Remark.** One could avoid the computations by observing that

$$\mathbf{r}(\theta) = < (1 - \cos 2\theta)/2, \sin 2\theta/2 > = < 1/2, 0 > + 1/2 < -\cos 2\theta, \sin 2\theta >$$

i.e. the curve is a circle centered at $(1/2, 0)$, with radius $1/2$. The curvature of a circle with radius $r$ is $1/r$, so $k(\theta) = 2$.

**Problem 6.** a) The domain of the function $f(x, y) = \ln(x^2 + x + y^2 - 1)$ is

$$D = \{(x, y) : x^2 + x + y^2 - 1 > 0\} = \{(x, y) : (x + 1/2)^2 + y^2 > 5/4\}$$

so $D$ is the outside of the circle with center $(-1/2, 0)$ and radius $\sqrt{5}/2$.

b) The level $k$ curve of the function $f(x, y) = e^{xy}$ is given by the equation $e^{xy} = k$. Thus the curves exist only for $k > 0$ and then they are given by $xy = \ln k$. For $k \in (0, 1)$ these curves are hyperbolas in the second and fourth quadrants, for $k > 1$ these are hyperbolas in first and third quadrants and for $k = 1$ the level curve is the union of the vertical and horizonal axes.

c) To see that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^3}{x^2 + y^2}$ does not exist let us approach $(0, 0)$ on the line $y = kx$. Then the limit will take form

$$\lim_{x\to 0} \frac{x^2 + k^3x^3}{x^2 + k^2x^2} = 1/(1 + k^2).$$

For $k = 0$ we get limit 1 and for $k = 1$ we get limit $1/2$. Since these limits have different values, the limit $\lim_{(x,y)\to(0,0)} \frac{x^2 + y^3}{x^2 + y^2}$ does not exist.

d) Let $f(x, y, z) = \frac{x^2 y}{x^2 + y^2 + z^2}$ for $(x, y, z) \neq (0, 0, 0)$ and $f(0, 0, 0) = a$. Since both the numerator and denominator are continuous functions, this function is continuous at all points except possibly the points where the denominator vanishes, i.e. the origin. It is continuous at the origin iff $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 y}{x^2 + y^2 + z^2} = a$. In order to compute the limit we use spherical coordinates $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho\cos\theta$. Thus

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 y}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho^3 \sin^3\phi \cos^2\theta \sin^2\theta}{\rho^2} = \lim_{\rho\to 0} \rho \sin^3\phi \cos^2\theta \sin^2\theta = 0$$

since $\rho$ approaches 0 and $\sin^3\phi\cos^2\theta\sin^2\theta$ is bounded between $-1$ and 1. Thus the function is continuous iff $a = 0$.

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**Problem 7.** a) Suppose that the acceleration and velocity of a smooth parametric curve $\mathbf{r}(t)$ are always orthogonal. Let $\mathbf{T}(t)$ be the unit tangent vector. Thus $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. We see that $\mathbf{a}(t) = \mathbf{v}'(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$. Differentiation of the equality $\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$ shows that $\mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$. Thus

$$0 = \mathbf{a}(t)\cdot\mathbf{v}(t) = (v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t))\cdot(v(t)\mathbf{T}(t)) = v'(t)v(t).$$

Since $v(t)$ is never 0, we conclude that $v'(t) = 0$ for all $t$, i.e. $v(t)$ is constant.

b) A plane curve $\mathbf{r}(t)$ has constant curvature $k > 0$. We may assume that the parametrization is arc-length. Thus $v(t) = 1$. It follows that $\mathbf{T}(t) = \mathbf{v}(t)$ and $k = k(t) = |\mathbf{a}(t)|$. Thus we have $\mathbf{v}(t)\cdot\mathbf{v}(t) = 1$ and $\mathbf{a}(t)\cdot\mathbf{a}(t) = k^2$. Differentiation of these identities yields $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$ and $\mathbf{a}(t)\cdot\mathbf{a}'(t) = 0$. Thus both $\mathbf{v}(t)$ and $\mathbf{a}'(t)$ are orthogonal to $\mathbf{a}(t)$, so the vectors $\mathbf{v}(t)$ and $\mathbf{a}'(t)$ are parallel. In other words, $\mathbf{a}'(t) = f(t)\mathbf{v}(t)$ for some scalar $f(t)$. Differentiation of $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$ gives $\mathbf{a}(t)\cdot\mathbf{a}(t) + \mathbf{v}(t)\cdot\mathbf{a}'(t) = 0$, i.e. $-k^2 = \mathbf{v}(t)\cdot\mathbf{a}'(t) = f(t)\mathbf{v}(t)\cdot\mathbf{v}(t) = f(t)$. Thus $0 = k^2\mathbf{v}(t) + \mathbf{a}'(t) = (k^2\mathbf{r}(t) + \mathbf{a}(t))'$. Therefore $k^2\mathbf{r}(t) + \mathbf{a}(t)$ is constant. Denote this constant $(x, y)$. Then $\mathbf{r}(t) =< x/k^2, y/k^2 > +\mathbf{a}(t)/k^2$. Since $\mathbf{a}(t)/k^2$ has constant length $1/k$, it follows that the curve is a circle with center $(x/k^2, y/k^2)$ and radius $1/k$.