Solutions to Exam II

Problem 1. a) Recall that the directional derivative $D_{\mathbf{u}}f(p)$ of the function f at the point p in the direction of the unit vector $\mathbf{u}$ is defined as follows

$$D_{\mathbf{u}}f(p) = \lim_{t\to 0} \frac{f(p+t\mathbf{u}) - f(p)}{t}.$$

In our case, $f(x,y) = (\sqrt[3]{x} + \sqrt[3]{y})^3$, $p = (0,0)$, and $\mathbf{u} =< 1/\sqrt{2}, 1/\sqrt{2} >$. Thus

$$D_{\mathbf{u}}f(p) = \lim_{t\to 0} \frac{\left(\sqrt[3]{\frac{t}{\sqrt{2}}} + \sqrt[3]{\frac{t}{\sqrt{2}}}\right)^3}{t} = \lim_{t\to 0} \frac{\left(2\sqrt[3]{\frac{t}{\sqrt{2}}}\right)^3}{t} = 8/\sqrt{2} = 4\sqrt{2}.$$

b) The same method as in a) shows that $\dfrac{\partial f}{\partial x}(0,0) = 1 = \dfrac{\partial f}{\partial y}(0,0)$, i.e. $\nabla f(0,0) =< 1, 1 >$. It follows that

$$\nabla f(0,0) \cdot \mathbf{u} = 2/\sqrt{2} \neq D_{\mathbf{u}}f(0,0),$$

which seems to contradict the theorem which states that $D_{\mathbf{u}}f(p) = \nabla f(p) \cdot \mathbf{u}$. However, there is no contradiction here, because the theorem assumes that f is differentiable at p and our function is not differentiable at $(0,0)$.

Problem 2. The surface is defined by $f(x,y,z) = 0$, where

$$f(x,y,z) = x^2 + y^2 + z^2 - xyz - 2.$$

Thus $\nabla f =< 2x - yz, 2y - xz, 2z - xy >$, so $\nabla f(1,1,0) =< 2, 2, -1 >$. The equation of the plane tangent to this surface at the point $(1,1,0)$ is then

$$2(x-1) + 2(y-1) - z = 0 \quad \text{or} \quad 2x + 2y - z = 4.$$

Problem 3. a) **Implicit Function Theorem:** Let $f(x_1, x_2, ..., x_n, x_{n+1})$ be a continuously differentiable function near a point $(a_1, ..., a_{n+1})$ such that $f(a_1, ..., a_{n+1}) = 0$ and $\dfrac{\partial f}{\partial x_{n+1}}(a_1, ..., a_{n+1}) \neq 0$. There exists a continuously differentiable function $g(x_1, ..., x_n)$, defined in some neighborhood U of the point $(a_1, ..., a_n)$, such that near the point $(a_1, ..., a_{n+1})$ the hyper-surface $f(x_1, x_2, ..., x_n, x_{n+1}) = 0$ coincides with the graph of the function $g(x_1, ..., x_n)$. In other words, $g(a_1, ..., a_n) = a_{n+1}$ and if $(x_1, ..., x_{n+1})$ is sufficiently close to $(a_1, ..., a_{n+1})$ then it satisfies the equation $f(x_1, x_2, ..., x_n, x_{n+1}) = 0$ iff $x_{n+1} = g(x_1, ..., x_n)$.

b) The Implicit Function Theorem implies that the surface

$$f(x, y, z) = x^2 + y^2 + z^2 - xyz - 2 = 0$$

is a graph of a function $z = g(x, y)$ near the point $(1, 1, 0)$, since $\frac{\partial f}{\partial z}(1, 1, 0) = -1 \neq 0$. From the equality $f(x, y, g(x, y)) = 0$ and the chain rule we get that

$$\frac{\partial g}{\partial x}(1,1) = \frac{-\frac{\partial f}{\partial x}(1,1,0)}{\frac{\partial f}{\partial z}(1,1,0)} = \frac{-2}{-1} = 2$$

and

$$\frac{\partial g}{\partial y}(1,1) = \frac{-\frac{\partial f}{\partial y}(1,1,0)}{\frac{\partial f}{\partial z}(1,1,0)} = \frac{-2}{-1} = 2$$

Thus the gradient $\nabla g(1, 1) =< 2, 2 >$.

c) Let $h(s, t) = F(x(s, t), y(s, t))$. The chain rule tells us that

$$\frac{\partial h}{\partial t}(a,b) = \frac{\partial F}{\partial x}(x(a,b), y(a,b))\frac{\partial x}{\partial t}(a,b) + \frac{\partial F}{\partial y}(x(a,b), y(a,b))\frac{\partial y}{\partial t}(a,b).$$

Taking $(a, b) = (0, 1)$ we see that

$$\frac{\partial h}{\partial t}(0,1) = 3 \cdot (-1) + 2 \cdot 1 = -1.$$

Problem 4. Recall that for continuously differentiable functions $f(x_1, ..., x_n)$, $g(x_1, ..., x_n)$, if f attains at a point $(a_1, ..., a_n)$ largest (smallest) value subject to $g(x_1, ..., x_n) = 0$ then either $\nabla g(a_1, ..., a_n) = 0$ or $\nabla f(a_1, ..., a_n) = \lambda \nabla g(a_1, ..., a_n)$ for some (unknown) constant λ. Thus points where f attains largest (smallest) value subject to $g = 0$ are either among the solutions to the system of equations $g(x_1, ..., x_n) = 0$, $\nabla g(x_1, ..., x_n) = 0$ or among the solutions to the system $g(x_1, ..., x_n) = 0$, $\nabla f(x_1, ..., x_n) = \lambda \nabla g(x_1, ..., x_n)$ (with unknowns $x_1, ..., x_n$ and λ).

In order to find largest and smallest values of the function $f(x, y) = x+y$ subject to the condition $g(x, y) = x^4 + 4xy + 2y^2 + 1 = 0$ we compute $\nabla f =< 1, 1 >$ and $\nabla g =< 4x^3 + 4y, 4x + 4y >$ (it is not hard to see that the equation $g(x, y) = 0$ describes a closed and bounded set, so f indeed attains its largest and smallest values subject to $g = 0$). We look first at the second system of equations, i.e. at

$$x^4 + 4xy + 2y^2 + 1 = 0, \quad 1 = \lambda(4x^3 + 4y), \quad 1 = \lambda(4x + 4y).$$

The last 2 equations imply that $\lambda \neq 0$ and $4x^3 = 4x$, i.e. $x = 0, 1$ or -1. Since $g(0, y) = 0$ has no solutions, we are left with two possibilities: either $x = 1$ or $x = -1$. If $x = 1$ then $g(1, y) = 2(y + 1)^2 = 0$ implies that $y = -1$. Similarly, for $x = -1$, we have $g(-1, y) = 2(y - 1)^2 = 0$, i.e. $y = 1$. So we have at most two solutions: $(1, -1)$ and $(-1, 1)$. But in fact neither one is a solution, since $\nabla g(1, -1) = 0 = \nabla g(-1, 1)$.

Both $(1, -1)$ and $(-1, 1)$ are solutions to the system $g(x, y) = 0$, $\nabla g(x, y) = 0$ and these are the only solutions to this system (which, explicitly, is

$$x^4 + 4xy + 2y^2 + 1 = 0, \quad 4x^3 + 4y = 0, \quad 4x + 4y = 0 \quad).$$

Thus the only points where f can attain largest (smallest) value are $(1, -1)$ and $(-1, 1)$. Note that $f(1, -1) = 0 = f(-1, 1)$. This says that the largest and smallest values coincide and are equal to 0.

Let us analyze this further. Since the largest and smallest values coincide, the function f must be constant on the set of solutions to $g(x, y) = 0$ and therefore f attains largest (or smallest) value at every solution to $g(x, y) = 0$. This shows that the only solutions to $g(x, y) = 0$ are $(1, -1)$ and $(-1, 1)$. One can show this easily directly, by observing that

$$g(x, y) = 2(x + y)^2 + (x^2 - 1)^2.$$

Exercise. Find the largest and smallest values of f subject to $x^4 + 4xy + 2y^2 - 1 = 0$.

Problem 5. Let $f(x, y) = x^2 - y^2 - x^2y^2$. Thus $\nabla f = < 2x - 2xy^2, -2y - 2yx^2 >$. To find critical points we solve the system $x - xy^2 = 0$, $-y - yx^2 = 0$. The second equation $-y(1 + x^2) = 0$ implies that $y = 0$ (since $1 + x^2$ is never 0). Now the first equation tells us that also $x = 0$. In other words, $(0, 0)$ is the only critical point of f. But it is not in the interior of D, so we do not need to worry about it (this point is on the boundary of D, so it will be considered when we investigate the boundary). It follows that both the largest and smallest values are attained at points on the boundary of D.

The boundary of D consists of two pieces: the interval $y = 0$, $-1 \leq x \leq 1$ and the semicircle $x^2 + y^2 = 1, y \geq 0$.

On the interval the function is $f(x, 0) = x^2$. Thus, on this interval, f attains largest value equal to 1 at $x = -1$ and $x = 1$ and smallest value equal to 0 at $x = 0$.

On the semicircle we have $y = \sqrt{1 - x^2}$ and our function equals $g(x) = f(x, \sqrt{1 - x^2}) = x^2 - (1 - x^2) - x^2(1 - x^2) = x^4 + x^2 - 1$, $x \in [-1, 1]$. Now $g'(x) = 4x^3 - 2x =$

$2x(2x^2-1)=0$ when $x=0$ or $x=1/\sqrt{2}$ or $x=-1/\sqrt{2}$. We have $g(-1)=g(1)=1$, $g(0)=-1$, $g(1/\sqrt{2})=g(-1/\sqrt{2})=-1/4$. Thus the largest value of f on the semi-circle is 1 and the smallest value equals -1.

Putting all the above together, we see that on D the function f has largest value equal to 1 and smallest value equal to -1.

Problem 6. We first compute all critical points of the function $f(x,y)=x^2+2xy^4-4xy^2$. We have $\nabla f=<2x+2y^4-4y^2, 8xy^3-8xy>$. Thus we need to solve the system of two equations: $2x+2y^4-4y^2=0$, $8xy^3-8xy=0$. The first equation can be written as $x=y^2(2-y^2)$ and the second is simply $xy(y^2-1)=0$. It follows that $y^2(2-y^2)y(y^2-1)=0$, which means that $y=0$, or $y^2=2$, or $y^2=1$. We have then five possibilities $y=-\sqrt{2},-1,0,1,\sqrt{2}$. From the first equation we compute the corresponding values of x: $0,1,0,1,0$. Thus the only critical points of f are $(0,-\sqrt{2})$, $(1,-1)$, $(0,0)$, $(1,1)$, $(0,\sqrt{2})$.

For each critical point (a,b) we need to compute the quantities

$$A=\frac{\partial^2 f}{\partial x^2}(a,b), B=\frac{\partial^2 f}{\partial x\partial y}(a,b), C=\frac{\partial^2 f}{\partial y^2}(a,b), \Delta=AC-B^2.$$

If $\Delta>0$ and $A>0$ then f has a local minimum at (a,b). If $\Delta>0$ and $A<0$ then f has a local maximum at (a,b). If $\Delta<0$ then (a,b) is a saddle point. When $\Delta=0$ further investigation is necessary to determine the type of the critical point.

Note that for $f=x^2+2xy^4-4xy^2$ we have

$$\frac{\partial^2 f}{\partial x^2}=2, \frac{\partial^2 f}{\partial x\partial y}=8y^3-8y, \frac{\partial^2 f}{\partial y^2}=8x(3y^2-1).$$

Now it is easy to see that for the points $(0,-\sqrt{2})$ and $(0,\sqrt{2})$ we have $\Delta=-128<0$, so these are saddle points.

For the points $(1,-1)$ and $(1,1)$ we have $\Delta=32$ and $A=2>0$, so f has a local minimum at $(1,-1)$ and at $(1,1)$.

Finally, for $(0,0)$ we get $\Delta=0$. To determine what type of critical point is $(0,0)$ note that, when $(0,0)$ is approached along the line $y=0$, our function $f(x,0)=x^2$ assumes positive values. Thus f assumes positive values in every neighborhood of $(0,0)$. On the other hand, when $(0,0)$ is approached along the curve $x=4y^2-3y^4$, our function

$$f(x,y)=x(x+2y^4-4y^2)=y^2(4-3y^2)(-y^4)=-y^6(4-3y^2)$$

is negative for all $y\in(0,2/\sqrt{3})$. Thus f also assumes negative values in every neighborhood of $(0,0)$. This shows that $(0,0)$ is a saddle point.

Problem 7. Consider two surfaces $f(x,y,z)=0$ and $g(x,y,z)=0$ and let (a,b,c) be a common point of these surfaces. If the gradients $\nabla f(a,b,c)$ and $\nabla g(a,b,c)$ are not parallel, then near the point (a,b,c) the surfaces intersect along a smooth curve (this is a version of Implicit Function Theorem) and the tangent line to this curve at the point (a,b,c) is simply the line of intersection of the tangent planes to both surfaces at (a,b,c). The normal vectors to these planes are $\nabla f(a,b,c)$ and $\nabla g(a,b,c)$. The vector $\nabla f(a,b,c)\times\nabla g(a,b,c)$ is orthogonal to both $\nabla f(a,b,c)$ and $\nabla g(a,b,c)$, so it is parallel to both tangent planes, hence also to the line of intersection. Thus this vector is tangent to the curve of intersection at the point (a,b,c)

In our case, $f(x,y,z)=x^4+y^4+z^4-3$ and $g(x,y,z)=x+y-2z$ and the point is $(1,1,1)$. Thus $\nabla f(1,1,1) =< 4,4,4 >$ and $\nabla g(1,1,1) =< 1,1,-2 >$. It follows that the vector $< 4,4,4 > \times < 1,1,-2 > = < -12,12,0 >$ is tangent to the curve of intersection at $(1,1,1)$.

Problem 8. The curve $x^2+3y^2=c$ can be parameterized by $x(t)=(\cos t)/\sqrt{c}$, $y(t)=(\sin t)/\sqrt{3c}$. Note that

$$\frac{\partial x}{\partial t}(t)=(-\sin t)/\sqrt{c}=-\sqrt{3}y(t) \quad \text{and} \quad \frac{\partial y}{\partial t}(t)=\frac{\cos t}{\sqrt{3c}}=\frac{x(t)}{\sqrt{3}}$$

In order to show that the function f is constant on the curve $x^2+3y^2=c$ it suffices to show that the function $g(t)=f(x(t),y(t))$ is constant. By the chain rule,

$$g'(t)=\frac{\partial f}{\partial x}(x(t),y(t))\frac{\partial x}{\partial t}(t)+\frac{\partial f}{\partial y}(x(t),y(t))\frac{\partial y}{\partial t}(t)=$$

$$\frac{\partial f}{\partial x}(x(t),y(t))(-\sqrt{3}y(t))+\frac{\partial f}{\partial y}(x(t),y(t))\frac{x(t)}{\sqrt{3}}=$$

$$\frac{1}{\sqrt{3}}\left(x(t)\frac{\partial f}{\partial y}(x(t),y(t))-3y(t)\frac{\partial f}{\partial x}(x(t),y(t))\right)=0$$

It follows that g is indeed constant.

It may seem that our solution heavily depends on the fact that an explicit parametrization of the curve $x^2+3y^2=c$ is known. But this is not the case. Let $(x(t),y(t))$ be an arbitrary parametrization of the curve, so that $x^2(t)+3y^2(t)=c$. Differentiation yields

$$2\frac{\partial x}{\partial t}(t)x(t)+6\frac{\partial y}{\partial t}(t)y(t)=0$$

This means that the vectors $<\frac{\partial x}{\partial t}(t),\frac{\partial y}{\partial t}(t)>$ and $<x(t),3y(t)>$ are orthogonal. The equality

$$x(t)\frac{\partial f}{\partial y}(x(t),y(t))-3y(t)\frac{\partial f}{\partial x}(x(t),y(t))=0$$

means that the vectors $< x(t), 3y(t) >$ and $< \frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) >$ are parallel. It follows that the vectors $< \frac{\partial x}{\partial t}(t), \frac{\partial y}{\partial t}(t) >$ and $< \frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) >$ are orthogonal, i.e.

$$\frac{\partial f}{\partial x}(x(t), y(t))\frac{\partial x}{\partial t}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\partial y}{\partial t}(t) = 0.$$

As we have seen, this equality means that the derivative of the function $g(t) = f(x(t), y(t))$ is 0, i.e. $g(t)$ is constant. Thus the function f is constant on the curve $x^2 + 3y^2 = c$.