Solutions to Exam I

Problem 1. a) The length of the vector $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ equals

$$|\mathbf{v}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}.$$

b) The vertices are $A(0,0)$, $B(0, 3+\sqrt{3})$, $C(3, \sqrt{3})$. Thus $\overrightarrow{AB} =< 0, 3+\sqrt{3} >$, $\overrightarrow{AC} =< 3, \sqrt{3} >$. It follows that $|\overrightarrow{AB}| = \sqrt{0^2 + (3+\sqrt{3})^2} = 3+\sqrt{3}$, $|\overrightarrow{AC}| = \sqrt{3^2 + (\sqrt{3})^2} = \sqrt{12}$ and

$$\cos A = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{(3+\sqrt{3})\sqrt{3}}{(3+\sqrt{3})\sqrt{12}} = 1/2,$$

i.e. $\measuredangle A = \pi/3$.

Similarly, $\overrightarrow{BA} =< 0, -3-\sqrt{3} >$, $\overrightarrow{BC} =< 3, -3 >$. It follows that $|\overrightarrow{BA}| = \sqrt{0^2 + (-3-\sqrt{3})^2} = 3+\sqrt{3}$, $|\overrightarrow{BC}| = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$ and

$$\cos B = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(-3-\sqrt{3})(-3)}{(3+\sqrt{3})3\sqrt{2}} = 1/\sqrt{2} = \sqrt{2}/2,$$

i.e. $\measuredangle B = \pi/4$.

Since the sum of all three angles in a triangle equals π, we get $\measuredangle C = \pi - \pi/3 - \pi/4 = 5\pi/12$.

c) Note that vectors $< a, b >$ and $< b, -a >$ are always orthogonal. Thus $< 4, -3 >$ is orthogonal to $< 3, 4 >$. Since $| < 4, -3 > | = 5$, the vector $< 4/5, -3/5 >$ is a unit vector orthogonal to $< 3, 4 >$.

d) Let $\mathbf{a} =< 1, 0, 1 >$ and $\mathbf{b} =< 1, 1, 0 >$. We are looking for vectors $\mathbf{u}$, $\mathbf{w}$ such that $\mathbf{u}$ is parallel to $\mathbf{a}$, $\mathbf{w}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b} = \mathbf{u} + \mathbf{w}$. Since $\mathbf{u}$ is parallel to

$\mathbf{a}$, we may write $\mathbf{u} = t\mathbf{a}$ for some scalar t. Taking the dot product of both sides of the equality $\mathbf{b} = t\mathbf{a} + \mathbf{w}$ with $\mathbf{a}$ we get

$$\mathbf{b} \cdot \mathbf{a} = t\mathbf{a} \cdot \mathbf{a} + \mathbf{w} \cdot \mathbf{a} = t\mathbf{a} \cdot \mathbf{a}$$

(since $\mathbf{w} \cdot \mathbf{a} = 0$). Thus $t = \mathbf{b} \cdot \mathbf{a}/\mathbf{a} \cdot \mathbf{a} = 1/2$, i.e. $\mathbf{u} = \mathbf{a}/2 =< 1/2, 0, 1/2 >$. Finally, $\mathbf{w} = \mathbf{b} - \mathbf{u} =< 1/2, 1, -1/2 >$.

Problem 2. a) $(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) =< 2, 1, -1 > \times < 1, -2, 3 >=$
$< 1 \cdot 3 - (-1)(-2), -(2 \cdot 3 - (-1) \cdot 1, 2 \cdot (-2) - 1 \cdot 1 >=< 1, -7, -5 >$. Alternatively, you could use the fact that the cross product is distributive with respect to addition.

b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 1 + (-1) = 1.$

c) The volume of the parallelepiped determined by vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ equals $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. For $\mathbf{u} =< 0, 0, 1 >$, $\mathbf{v} =< 1, 0, 1 >$, $\mathbf{w} =< 1, 1, 1 >$ we have $\mathbf{v} \times \mathbf{w} =< -1, 0, 1 >$ and $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1$, so the volume equals 1 .

Alternatively, the volume of the parallelepiped is the absolute value of the determinant

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1.$$

d) The area of a triangle with vertices A, B, C equals half of the area of the parallelogram determined by the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so it is equal to $|\overrightarrow{AB} \times \overrightarrow{AC}|/2$. In the problem, $A(0, 0, 0)$, $B(1, 0, 1)$, $C(1, 1, 1)$, so $\overrightarrow{AB} =< 1, 0, 1 >$, $\overrightarrow{AC} =< 1, 1, 1 >$ and the area equals

$$| < 1, 0, 1 > \times < 1, 1, 1 > |/2 = | < -1, 0, 1 > |/2 = \sqrt{2}/2.$$

Problem 3. a) $x^2 + y^2 + z^2 = x - y + z$ may be written as

$$(x - 1/2)^2 + (y + 1/2)^2 + (z - 1/2)^2 = 3/4$$

so this equation describes the sphere with center $(1/2, -1/2, 1/2)$ and radius $\sqrt{3}/2$.

b) Adding the equations $2x - y - z = 0$ and $x - 2y + z = 0$ yields $3x - 3y = 0$, i.e. $x = y$. Thus $z = 2x - y = 2x - x = x$, i.e. $x = y = z$. It follows that the parametric equation is $x = t, y = t, z = t$ and the symmetric equation is $x = y = z$.

Alternatively, first find two distinct points belonging to both planes. For example, $A(0, 0, 0)$ and $B(1, 1, 1)$ work. The line of intesection is the line passing through A and parallel to the vector $\overrightarrow{AB} =< 1, 1, 1 >$. It follows that the parametric equation is $x = t, y = t, z = t$ and the symmetric equation is $x = y = z$.

c) The plane containing points A, B, C is orthogonal to a vector orthogonal to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$, i.e. it is orthogonal to $\overrightarrow{AB} \times \overrightarrow{AC}$. In the problem, $A(1, 0, 1)$, $B(0, 1, 1)$, $C(1, 1, 0)$. Thus $\overrightarrow{AB} \times \overrightarrow{AC} =< -1, 1, 0 > \times < 0, 1, -1 >=< -1, -1, -1 >$. Thus we want an equation of the plane passing through $(1, 0, 1)$ and orthogonal to $< -1, -1, -1 >$, which is $-(x - 1) - y - (z - 1) = 0$, i.e. $x + y + z = 2$.

Problem 4. a) The curvature of a circle of radius R is $1/R$.

b) We have $\mathbf{r}(t) =< 2t - \sin 2t, -\cos 2t, 4 \sin t >$.
The velocity $\mathbf{v}(t) = \mathbf{r}'(t) =< 2 - 2\cos 2t, 2 \sin 2t, 4 \cos t >$.
Thus $\mathbf{v}(\pi/2) =< 4, 0, 0 >$.
The speed $v(\pi/2) = 4$
(We do not need a general formula for speed but it can be computed as follows $v(t) = |\mathbf{v}(t)| = \sqrt{(2 - 2\cos 2t)^2 + (2 \sin 2t)^2 + (4 \cos t)^2} =$

$$\sqrt{4 - 8\cos 2t + 4\cos^2 2t + 4\sin^2 2t + 16\cos^2 t} =$$

$$\sqrt{4 + 4(\cos^2 2t + \sin^2 2t) - 8(2\cos^2 t - 1) + 16\cos^2 t} = \sqrt{4 + 4 + 8} = 4.$$

We used the identity $\cos 2t = 2\cos^2 t - 1$.).

The acceleration $\mathbf{a}(t) = \mathbf{v}'(t) =< 4 \sin 2t, 4 \cos 2t, -4 \sin t >$ so $\mathbf{a}(\pi/2) =< 0, -4, -4 >$.
The unit tangent vector $\mathbf{T}(\pi/2) = \mathbf{v}(\pi/2t)/v(\pi/2) =< 1, 0, 0 >$.
(In general, $\mathbf{T}(t) = \mathbf{v}(t)/v(t) =< (1 - \cos 2t)/2, \sin 2t/2, \cos t >$, but we do not need it.).

The curvature $k(\pi/2) = |\mathbf{v}(\pi/2) \times \mathbf{a}(\pi/2)|/v^3(\pi/2) =$
$| < 4, 0, 0 > \times < 0, -4, -4 > |/4^3 = | < 0, 16, -16 > |/4^3 = \sqrt{2}/4$.

(The computation of $k(t)$ in general from the formula $k(t) = |\mathbf{v}(t)\times\mathbf{a}(t)|/v^3(t)$ is a bit complicated and requires some trigonometric identities. But recall that $|\mathbf{v}(t)\times\mathbf{a}(t)|$ is the area of the parallelogram determined by $\mathbf{v}(t)$ and $|\mathbf{a}(t)|$. Since the speed is constant, the velocity and acceleration are orthogonal, so the parallelogram is a rectangle and therefore its area is $|\mathbf{v}(t)||\mathbf{a}(t)| = 4\sqrt{16+16\sin^2 t} = 16\sqrt{1+\sin^2 t}$. Thus $k(t) = \sqrt{1+\sin^2 t}/4$.

Another method is to use the formula $k(t) = |\mathbf{T}'(t)|/v(t) = \sqrt{1+\sin^2 t}/4$.).

The unit normal vector is computed from the formula

$$\mathbf{a}(t) = k(t)v(t)^2\mathbf{N}(t) + \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{v(t)}\mathbf{T}(t).$$

For $t = \pi/2$ we get $\mathbf{v}(\pi/2)\cdot\mathbf{a}(\pi/2) = 0$ and

$$< 0, -4, -4 >= \frac{\sqrt{2}}{2}4^2\mathbf{N}(\pi/2)$$

so $\mathbf{N}(\pi/2) =< 0, -\sqrt{2}/2, \sqrt{2}/2 >$.
(We have seen that the acceleration and velocity are orthogonal for all t, so $\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$ and $\mathbf{a}(t) = k(t)v(t)^2\mathbf{N}(t)$, i.e.

$$\mathbf{N}(t) = \frac{\mathbf{a}(t)}{k(t)v(t)^2} =< \frac{\sin 2t}{\sqrt{1+\sin^2 t}}, \frac{\cos 2t}{\sqrt{1+\sin^2 t}}, \frac{-\sin t}{\sqrt{1+\sin^2 t}} > .$$

Alternatively, $\mathbf{N}(t)$ can be computed from the formula

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{< \sin 2t, \cos 2t, -\sin t >}{\sqrt{\sin^2 2t + \cos^2 2t + \sin^2 t}} =< \frac{\sin 2t}{\sqrt{1+\sin^2 t}}, \frac{\cos 2t}{\sqrt{1+\sin^2 t}}, \frac{-\sin t}{\sqrt{1+\sin^2 t}} > .)$$

c) The velocity of the parametric curve $\mathbf{r}(t) =< t\sin t, t\cos t, \frac{2\sqrt{2}}{3}t^{3/2} >$ (note that it is defined only for $t \geq 0$) equals

$$\mathbf{v}(t) =< \sin t + t\cos t, \cos t - t\sin t, \sqrt{2t} > .$$

The speed $v(t) = \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + (\sqrt{2t})^2} =$

$\sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + 2t} = \sqrt{1+t^2+2t} = 1+t.$

The length of the curve between $\mathbf{r}(0)$ and $\mathbf{r}(t)$ is

$$s(t) = \int_0^t v(u)du = \int_0^t (1+u)du = t + t^2/2.$$

From the equation $s = t + t^2/2$ we get $1 + 2s = 1 + 2t + t^2 = (1 + t)^2$, hence $t = \sqrt{2s+1} - 1$. The arc-length parametrization is then

$$\mathbf{r}(s) =< (\sqrt{2s+1}-1)\sin(\sqrt{2s+1}-1), (\sqrt{2s+1}-1)\cos(\sqrt{2s+1}-1), \frac{2\sqrt{2}}{3}(\sqrt{2s+1}-1)^{3/2} > .$$

d) A particle moves in the space with acceleration $\mathbf{a}(t) =< 2, 6t, 12t^2 >$. The velocity of the particle equals

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt =< 2t + c_1, 3t^2 + c_2, 4t^3 + c_3 >$$

for some constants c_1, c_2, c_3. The condition $\mathbf{v}(1) =< 3, 4, 5 >$ implies that $c_1 = c_2 = c_3 = 1$, i.e. $\mathbf{v}(t) =< 2t + 1, 3t^2 + 1, 4t^3 + 1 >$. The position of the particle equals

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt =< t^2 + t + d_1, t^3 + t + d_2, t^4 + t + d_3 > .$$

The condition $\mathbf{r}(1) =< 3, 2, 2 >$ implies that $d_1 = 1$, $d_2 = d_3 = 0$, i.e. $\mathbf{r}(t) =< t^2 + t + 1, t^3 + t, t^4 + t >$. Thus, at $t = 0$ the particle is at the point $(1, 0, 0)$.

Problem 5. a) The cylindrical coordinates of the point $(1, 1, \sqrt{6})$ are (r, θ, z), where $r^2 = 1^2 + 1^2 = 2$, $\tan\theta = 1/1 = 1$, $z = \sqrt{6}$. Thus $\theta = \pi/4$ and the cylindrical coordinates are $(\sqrt{2}, \pi/4, \sqrt{6})$.

The spherical coordinates of this point are (ρ, ϕ, θ), where $\theta = \pi/4$ is the same as for the cylindrical coordinates, $\rho^2 = 1^2 + 1^2 + \sqrt{6}^2 = 8$ and $\cos\phi = \sqrt{6}/\rho = \sqrt{6}/\sqrt{8} = \sqrt{3}/2$. Thus $\phi = \pi/6$ and the spherical coordinates are $(2\sqrt{2}, \pi/6, \pi/4)$.

b) The point whose cylindrical coordinates are $(1, \pi/6, 1)$ has Cartesian coordinates $(\cos\pi/6, \sin\pi/6, 1)$. The spherical coordinates are (ρ, ϕ, θ), where $\theta = \pi/6$ (same as for cylindrical coordinates), $\rho^2 = \cos^2\pi/6 + \sin^2\pi/6 + 1 = 2$ and $\cos\phi = 1/\rho = 1/\sqrt{2} = \sqrt{2}/2$. Thus $\phi = \pi/4$ and the spherical coordinates are $(\sqrt{2}, \pi/4, \pi/6)$.

c) A plane curve in polar coordinates has equation $r = \cos\theta$. Since $x = r\cos\theta = \cos^2\theta$, $y = r\sin\theta = \cos\theta\sin\theta$, the curve has parametric equation $\mathbf{r}(\theta) =< \cos^2\theta, \sin\theta\cos\theta >$ in Cartesian coordinates. The velocity is $\mathbf{v}(\theta) =< -2\sin\theta\cos\theta, \cos^2\theta - \sin^2\theta >$. The speed $v(\theta) = \sqrt{(-2\sin\theta\cos\theta)^2 + (\cos^2\theta - \sin^2\theta)^2} = \sqrt{(\cos^2\theta + \sin^2\theta)^2} = 1$. The acceleration $\mathbf{a}(\theta) =< -2(\cos^2\theta - \sin^2\theta), -4\sin\theta\cos\theta >$. Since the parametrization $\mathbf{r}(\theta)$ is a natural (arc-length) parametrization, we have $\mathbf{T}(\theta) = \mathbf{v}(\theta)$ and

$k(\theta) = |\mathbf{T}'(\theta)| = |\mathbf{a}(\theta)|$. Thus

$$k(\theta) = \sqrt{[(-2)(\cos^2\theta - \sin^2\theta)]^2 + (-4\sin\theta\cos\theta)^2} = \sqrt{4(\cos^2\theta + \sin^2\theta)^2} = 2.$$

Remark. The computation simplify significantly when the formulas $\sin 2x = 2\sin x\cos x$, $\cos 2x = \cos^2 x - \sin^2 x$ are used.

Remark. One could avoid the computations by observing that

$$\mathbf{r}(\theta) =< (1-\cos 2\theta)/2, \sin 2\theta/2 >=< 1/2, 0 > +1/2 < -\cos 2\theta, \sin 2\theta >$$

i.e. the curve is a circle centered at $(1/2, 0)$, with radius $1/2$. The curvature of a circle with radius r is $1/r$, so $k(\theta) = 2$.

Problem 6. a) The intersection with the XY plane is a curve with equation $y^2 - 4x^2 = 100$ so it is a hiperbola.

b) The intersection with the XZ plane has equation $-4x^2 - 25z^2 = 100$. Since the left hand side is never positive, the equation describes the empty set.

c) The intersection with the YZ plane is a curve with equation $y^2 - 25z^2 = 100$ so it is a hiperbola.

The surface is a hyperboloid of 2 sheets.

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Problem 7. Suppose that the acceleration and velocity of a smooth parametric curve $\mathbf{r}(t)$ are always orthogonal. Let $\mathbf{T}(t)$ be the unit tangent vector. Thus $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. We see that $\mathbf{a}(t) = \mathbf{v}'(t) = v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t)$. Differentiation of the equality $\mathbf{T}(t)\cdot\mathbf{T}(t) = 1$ shows that $\mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$. Thus

$$0 = \mathbf{a}(t)\cdot\mathbf{v}(t) = (v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t))\cdot(v(t)\mathbf{T}(t)) = v'(t)v(t).$$

Since $v(t)$ is never 0, we conclude that $v'(t) = 0$ for all t, i.e. $v(t)$ is constant.