Solutions to Exam II

Problem 1. a) To see that the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^3}{x^2+y^2}$ does not exist let us approach $(0,0)$ along the line $y = kx$. Then the limit will take form

$$\lim_{x\to 0} \frac{x^2+k^3x^3}{x^2+k^2x^2} = 1/(1+k^2).$$

For $k = 0$ we get limit 1 and for $k = 1$ we get limit $1/2$. Since these limits have different values, the limit $\lim_{(x,y)\to(0,0)} \dfrac{x^2+y^3}{x^2+y^2}$ does not exist.

b) Let $f(x,y,z) = \dfrac{x^2y}{x^2+y^2+z^2}$ for $(x,y,z) \neq (0,0,0)$ and $f(0,0,0) = a$. Since both the numerator and denominator are continuous functions, this function is continuous at all points except possibly the points where the denominator vanishes, i.e. the origin. It is continuous at the origin iff $\lim_{(x,y,z)\to(0,0,0)} \dfrac{x^2y}{x^2+y^2+z^2} = a$. In order to compute the limit we use spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\theta$. Thus

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{\rho^3\sin^3\phi\cos^2\theta\sin^2\theta}{\rho^2} = \lim_{\rho\to 0} \rho\sin^3\phi\cos^2\theta\sin^2\theta = 0$$

since ρ approaches 0 and $\sin^3\phi\cos^2\theta\sin^2\theta$ is bounded between -1 and 1. Thus the function is continuous iff $a = 0$.

Problem 2. a) First we compute the partial derivatives of g:

$$\frac{\partial g}{\partial x}(x,y) = 4x^3 + 4y, \quad \frac{\partial g}{\partial y}(x,y) = 4x + 4y.$$

Thus (x,y) is a critical point of g iff $4x^3 + 4y = 0$ and $4x + 4y$. In other words, $y = -x$ and $x^3 - x = 0$. The second equation holds iff $x = 0$, or $x = -1$, or $x = 1$ and then $y = 0$, $y = 1$, $y = -1$ respectively. Thus the critical points are $(0,0)$, $(-1,1)$, $(1,-1)$. We know that if a function attains its smallest value in an interior point of its domain then this point must be a critical point. Thus, the smallest value of g is attained at some of its critical points. Since $g(0,0) = -1$, $g(-1,1) = -2 = g(1,-1)$, we see that the smallest value if g is -2 and it is attained at two points: $(-1, 1)$ and $(1, -1)$.

b) Recall that if a function f has continuous second order mixed partial derivatives, then they must be equal. If a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x}(x, y) = \sin x - \cos y \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x \cos y$$

existed, its mixed second order partial derivatives would be

$$\frac{\partial^2 f}{\partial y \partial x}(x, y) = \sin y \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}(x, y) = \cos y.$$

Thus f would have continuous second order mixed partial derivatives which are different, a contradiction. It follows that such function f does not exist.

Problem 3. a) The surface is defined by $f(x, y, z) = 0$, where

$$f(x, y, z) = x^2 + y^2 + z^2 - xyz - 2.$$

Thus $\nabla f =< 2x - yz, 2y - xz, 2z - xy >$, so $\nabla f(1, 1, 0) =< 2, 2, -1 >$. The equation of the plane tangent to this surface at the point $(1, 1, 0)$ is then

$$2(x - 1) + 2(y - 1) - z = 0 \ \text{ i.e. } \ 2x + 2y - z = 4.$$

b) Consider two surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ and let (a, b, c) be a common point of these surfaces. If the gradients $\nabla f(a, b, c)$ and $\nabla g(a, b, c)$ are not parallel, then near the point (a, b, c) the surfaces intersect along a smooth curve (this is a version of Implicit Function Theorem) and the tangent line to this curve at the point (a, b, c) is simply the line of intersection of the tangent planes to both surfaces at (a, b, c). The normal vectors to these planes are $\nabla f(a, b, c)$ and $\nabla g(a, b, c)$. The vector $\nabla f(a, b, c) \times \nabla g(a, b, c)$ is orthogonal to both $\nabla f(a, b, c)$ and $\nabla g(a, b, c)$, so it is parallel to both tangent planes, hence also to the line of intersection. Thus this vector is tangent to the curve of intersection at the point (a, b, c)

In our case, $f(x, y, z) = x^4 + y^4 + z^4 - 3$ and $g(x, y, z) = x + y - 2z$ and the point is $(1, 1, 1)$. Thus $\nabla f(1, 1, 1) =< 4, 4, 4 >$ and $\nabla g(1, 1, 1) =< 1, 1, -2 >$. It follows that the vector $< 4, 4, 4 > \times < 1, 1, -2 >=< -12, 12, 0 >$ is tangent to the curve of intersection at $(1, 1, 1)$. The tangent line has then parametric equation

$$x = 1 - 12t, \quad y = 1 + 12t, \quad z = 1.$$

Problem 4. a) The Implicit Function Theorem implies that the surface

$$f(x,y,z) = x^2 + y^2 + z^2 - xyz - 2 = 0$$

is a graph of a function $z = g(x,y)$ near the point $(1,1,0)$. This means that $f(x,y,g(x,y)) = 0$. Differentiating this equality and applying the chain rule we get that

$$\frac{\partial f}{\partial x}(x,y,g(x,y)) + \frac{\partial f}{\partial z}(x,y,g(x,y))\frac{\partial g}{\partial x}(x,y) = 0$$

and

$$\frac{\partial f}{\partial y}(x,y,g(x,y)) + \frac{\partial f}{\partial z}(x,y,g(x,y))\frac{\partial g}{\partial y}(x,y) = 0.$$

Thus

$$\frac{\partial g}{\partial x}(1,1) = \frac{-\frac{\partial f}{\partial x}(1,1,0)}{\frac{\partial f}{\partial z}(1,1,0)} = \frac{-2}{-1} = 2$$

and

$$\frac{\partial g}{\partial y}(1,1) = \frac{-\frac{\partial f}{\partial y}(1,1,0)}{\frac{\partial f}{\partial z}(1,1,0)} = \frac{-2}{-1} = 2$$

It follows that the gradient $\nabla g(1,1) =< 2, 2 >$.

b) Recall the **Implicit Function Theorem:** Let $f(x_1, x_2, ..., x_n, x_{n+1})$ be a continuously differentiable function near a point $(a_1, ..., a_{n+1})$ such that $f(a_1, ..., a_{n+1}) = 0$ and $\frac{\partial f}{\partial x_{n+1}}(a_1, ..., a_{n+1}) \neq 0$. There exists a continuously differentiable function $g(x_1, ..., x_n)$, defined in some neighborhood U of the point $(a_1, ..., a_n)$, such that near the point $(a_1, ..., a_{n+1})$ the hyper-surface $f(x_1, x_2, ..., x_n, x_{n+1}) = 0$ coincides with the graph of the function $g(x_1, ..., x_n)$. In other words, $g(a_1, ..., a_n) = a_{n+1}$ and if $(x_1, ..., x_{n+1})$ is sufficiently close to $(a_1, ..., a_{n+1})$ then it satisfies the equation $f(x_1, x_2, ..., x_n, x_{n+1}) = 0$ iff $x_{n+1} = g(x_1, ..., x_n)$.

In our case, $f(x,y,z) = x^2+y^2+z^2-xyz-2$ so we need to check that f is continuously differentiable and $\frac{\partial f}{\partial z}(1,1,0) \neq 0$. This is indeed true, since $\frac{\partial f}{\partial z}(1,1,0) = -1$.

c) Let $h(s,t) = F(x(s,t), y(s,t))$. The chain rule tells us that

$$\frac{\partial h}{\partial t}(a,b) = \frac{\partial F}{\partial x}(x(a,b), y(a,b))\frac{\partial x}{\partial t}(a,b) + \frac{\partial F}{\partial y}(x(a,b), y(a,b))\frac{\partial y}{\partial t}(a,b).$$

Taking $(a,b) = (0,1)$ we see that

$$\frac{\partial h}{\partial t}(0,1) = 3 \cdot (-1) + 2 \cdot 1 = -1.$$

Problem 5. Let $f(x,y) = x^2 - y^2 - x^2y^2$. Thus $\nabla f =< 2x - 2xy^2, -2y - 2yx^2 >$. To find critical points we solve the system $x - xy^2 = 0$, $-y - yx^2 = 0$. The second equation $-y(1+x^2)=0$ implies that $y=0$ (since $1+x^2$ is never 0). Now the first equation tells us that also $x=0$. In other words, $(0,0)$ is the only critical point of f. But it is not in the interior of D, so we do not need to worry about it (this point is on the boundary of D, so it will be considered when we investigate the boundary). It follows that both the largest and smallest values are attained at points on the boundary of D.

The boundary of D consists of two pieces: the interval $y=0$, $-1 \leq x \leq 1$ and the semicircle $x^2+y^2=1, y \geq 0$.

On the interval the function is $f(x,0)=x^2$. Thus, on this interval, f attains largest value equal to 1 at $x=-1$ and $x=1$ and smallest value equal to 0 at $x=0$.

We can parametrize the semicircle by $x=\cos t$, $y=\sin t$, $t \in [0,\pi]$. Thus $f(x,y)=\cos^2 t-\sin^2 t-\cos^2 t\sin^2 t=h(t)$. We have $h(0)=1$, $h(\pi)=1$ and

$$h'(t)=-2\cos t\sin t-2\sin t\cos t+2\cos t\sin^3 t-2\sin t\cos^3 t=2\sin t\cos t(-2+\sin^2 t-\cos^2 t)$$

(you can simplify a bit the computations by using basic trigonometry). We see that $h'(t)=0$ iff either $\sin t=0$ or $\cos t=0$, or $(-2+\sin^2 t-\cos^2 t)=0$. The first and last of these possibilities can not happen for $t \in (0,\pi)$, so $\cos t=0$ and $t=\pi/2$. Since $h(\pi/2)=-1$, we see that the largest value of f on the semicircle is 1 and the smallest value equals -1.

Alternatively, on the semicircle we have $y=\sqrt{1-x^2}$ and our function equals $g(x)=f(x,\sqrt{1-x^2})=x^2-(1-x^2)-x^2(1-x^2)=x^4+x^2-1$, $x \in [-1,1]$. Now $g'(x)=4x^3-2x=2x(2x^2-1)=0$ when $x=0$ or $x=1/\sqrt{2}$ or $x=-1/\sqrt{2}$. We have $g(-1)=g(1)=1$, $g(0)=-1$, $g(1/\sqrt{2})=g(-1/\sqrt{2})=-1/4$. Thus the largest value of f on the semicircle is 1 and the smallest value equals -1.

Putting all the above together, we see that on D the function f has largest value equal to 1 and smallest value equal to -1.

Problem 6. Recall that for continuously differentiable functions $f(x_1,...,x_n)$, $g(x_1,...,x_n)$, if f attains at a point $(a_1,...,a_n)$ largest (smallest) value subject to $g(x_1,...,x_n)=0$ then either $\nabla g(a_1,...,a_n)=0$ or $\nabla f(a_1,...,a_n)=\lambda\nabla g(a_1,...,a_n)$ for some (unknown) constant λ. Thus points where f attains largest (smallest) value subject to $g=0$ are either among the solutions to the system of equations $g(x_1,...,x_n)=0$, $\nabla g(x_1,...,x_n)=0$ or among the solutions to the system $g(x_1,...,x_n)=0$, $\nabla f(x_1,...,x_n)=\lambda\nabla g(x_1,...,x_n)$ (with unknowns $x_1,...,x_n$ and λ).

In order to find largest and smallest values of the function $f(x,y)=x+y$ subject to the condition $g(x,y)=x^4+4xy+2y^2-1=0$ we compute $\nabla f=<1,1>$

and $\nabla g = < 4x^3 + 4y, 4x + 4y >$ (it is not hard to see that the equation $g(x, y) = 0$ describes a closed and bounded set, so f indeed attains its largest and smallest values subject to $g = 0$). Note that in Problem 2a) we have seen that $\nabla g(x, y) = 0$ iff (x, y) is one of $(0, 0)$, $(-1, 1)$, $(1, -1)$. None of these three points satisfies $g(x, y) = 0$, so we only need to consider the second system of equations, i.e.

$$x^4 + 4xy + 2y^2 - 1 = 0, \quad 1 = \lambda(4x^3 + 4y), \quad 1 = \lambda(4x + 4y).$$

The last 2 equations imply that $4x^3 = 4x$, i.e. $x = 0, 1$ or -1. For $x = 0$ we get $g(0, y) = 2y^2 - 1 = 0$, so $y = -1/\sqrt{2}$ or $y = 1/\sqrt{2}$. If $x = 1$ then $g(1, y) = 4y + 2y^2 = 0$ implies that $y = 0$ or $y = -2$. Similarly, for $x = -1$, we have $g(-1, y) = -4y + 2y^2 = 0$, i.e. $y = 0$ or $y = 2$. So we have 6 points where the largest and smallest values can be attained: $(0, -1/\sqrt{2})$, $(0, 1/\sqrt{2})$, $(1, 0)$, $(1, -2)$, $(-1, 0)$, $(-1, 2)$. Since $f(0, -1/\sqrt{2}) = -1/\sqrt{2}$, $f(0, 1/\sqrt{2}) = 1/\sqrt{2}$, $f(1, 0) = 1$, $f(1, -2) = -1$, $f(-1, 0) = -1$, $f(-1, 2) = 1$, we see that the largest value of f is 1 and the smallest value is -1.

Problem 7. The curve $x^2 + 3y^2 = c$ can be parameterized by $x(t) = (\cos t)/\sqrt{c}$, $y(t) = (\sin t)/\sqrt{3c}$. Note that

$$\frac{\partial x}{\partial t}(t) = (-\sin t)/\sqrt{c} = -\sqrt{3}y(t) \quad \text{and} \quad \frac{\partial y}{\partial t}(t) = \frac{\cos t}{\sqrt{3c}} = \frac{x(t)}{\sqrt{3}}$$

In order to show that the function f is constant on the curve $x^2 + 3y^2 = c$ it suffices to show that the function $g(t) = f(x(t), y(t))$ is constant. By the chain rule,

$$g'(t) = \frac{\partial f}{\partial x}(x(t), y(t))\frac{\partial x}{\partial t}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\partial y}{\partial t}(t) =$$

$$\frac{\partial f}{\partial x}(x(t), y(t))(-\sqrt{3}y(t)) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{x(t)}{\sqrt{3}} =$$

$$\frac{1}{\sqrt{3}}\left(x(t)\frac{\partial f}{\partial y}(x(t), y(t)) - 3y(t)\frac{\partial f}{\partial x}(x(t), y(t))\right) = 0$$

It follows that g is indeed constant.

It may seem that our solution heavily depends on the fact that an explicit parametrization of the curve $x^2 + 3y^2 = c$ is known. But this is not the case. Let $(x(t), y(t))$ be an arbitrary parametrization of the curve, so that $x^2(t) + 3y^2(t) = c$. Differentiation yields

$$2\frac{\partial x}{\partial t}(t)x(t) + 6\frac{\partial y}{\partial t}(t)y(t) = 0$$

This means that the vectors $< \frac{\partial x}{\partial t}(t), \frac{\partial y}{\partial t}(t) >$ and $< x(t), 3y(t) >$ are orthogonal. The equality

$$x(t)\frac{\partial f}{\partial y}(x(t), y(t)) - 3y(t)\frac{\partial f}{\partial x}(x(t), y(t)) = 0$$

means that the vectors $< x(t), 3y(t) >$ and $< \frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) >$ are parallel. It follows that the vectors $< \frac{\partial x}{\partial t}(t), \frac{\partial y}{\partial t}(t) >$ and $< \frac{\partial f}{\partial x}(x(t), y(t)), \frac{\partial f}{\partial y}(x(t), y(t)) >$ are orthogonal, i.e.

$$\frac{\partial f}{\partial x}(x(t), y(t))\frac{\partial x}{\partial t}(t) + \frac{\partial f}{\partial y}(x(t), y(t))\frac{\partial y}{\partial t}(t) = 0.$$

As we have seen, this equality means that the derivative of the function $g(t) = f(x(t), y(t))$ is 0, i.e. $g(t)$ is constant. Thus the function f is constant on the curve $x^2 + 3y^2 = c$.