Solutions to Exam III

Solution to Problem 1. We first compute all critical points of the function $f(x,y) = x^2 + 2xy^4 - 4xy^2$ (even though they are given to us in the statement of the problem). We have $\nabla f =< 2x + 2y^4 - 4y^2, 8xy^3 - 8xy >$. Thus we need to solve the system of two equations: $2x + 2y^4 - 4y^2 = 0$, $8xy^3 - 8xy = 0$. The first equation can be written as $x = y^2(2 - y^2)$ and the second is simply $xy(y^2 - 1) = 0$. It follows that $y^2(2-y^2)y(y^2-1) = 0$, which means that $y = 0$, or $y^2 = 2$, or $y^2 = 1$. We have then five possibilities $y = -\sqrt{2}, -1, 0, 1, \sqrt{2}$. From the first equation we compute the corresponding values of x: $0, 1, 0, 1, 0$. Thus the only critical points of f are $(0, -\sqrt{2})$, $(1, -1)$, $(0, 0)$, $(1, 1)$, $(0, \sqrt{2})$.

For each critical point (a, b) we need to compute the quantities

$$A = \frac{\partial^2 f}{\partial x^2}(a,b), B = \frac{\partial^2 f}{\partial x \partial y}(a,b), C = \frac{\partial^2 f}{\partial y^2}(a,b), \Delta = AC - B^2.$$

If $\Delta > 0$ and $A > 0$ then f has a local minimum at (a, b). If $\Delta > 0$ and $A < 0$ then f has a local maximum at (a, b). If $\Delta < 0$ then (a, b) is a saddle point. Note that for $f = x^2 + 2xy^4 - 4xy^2$ we have

$$\frac{\partial^2 f}{\partial x^2} = 2, \frac{\partial^2 f}{\partial x \partial y} = 8y^3 - 8y, \frac{\partial^2 f}{\partial y^2} = 8x(3y^2 - 1).$$

Now it is easy to see that for the points $(0, -\sqrt{2})$ and $(0, \sqrt{2})$ we have $\Delta = -128 < 0$, so these are saddle points.

For the points $(1, -1)$ and $(1, 1)$ we have $\Delta = 32$ and $A = 2 > 0$, so f has a local minimum at $(1, -1)$ and at $(1, 1)$.

Finally, for $(0, 0)$ we get $\Delta = 0$. To determine what type of critical point is $(0, 0)$ note that, when $(0, 0)$ is approached along the line $y = 0$, our function $f(x, 0) = x^2$ assumes positive values. Thus f assumes positive values in every neighborhood of $(0, 0)$. On the other hand, when $(0, 0)$ is approached along the parabola $y^2 = x$, our function

$$f(x,y) = x(x + 2y^4 - 4y^2) = x(x + 2x^2 - 4x) = x^2(-3 + 2x)$$

is negative for all $x < 3/2$. Thus f also assumes negative values in every neighborhood of $(0,0)$. This shows that $(0,0)$ is a saddle point.

Solution to Problem 2. a) The region bounded by the parabola $y = x^2 - 2$, the line $y = x + 1$ and the vertical lines $x = -1$ and $x = 2$ is vertically simple. Thus

$$\int\int_R \frac{1}{3 + x - x^2}\, \mathrm{d}x\, \mathrm{d}y = \int_{-1}^{2}\int_{x^2-2}^{x+1} \frac{1}{3 + x - x^2}\, \mathrm{d}y\, \mathrm{d}x = \int_{-1}^{2} \frac{1}{3 + x - x^2}\left(\int_{x^2-2}^{x+1} \mathrm{d}y\right) \mathrm{d}x$$

$$= \int_{-1}^{2} \frac{1}{3 + x - x^2}\big((x+1) - (x^2 - 2)\big)\, \mathrm{d}x = \int_{-1}^{2} \mathrm{d}x = 3$$

b) The solid T bounded by the parabolic cylinder $y = 1 - x^2$ and the planes $y = 0$, $z = 1$, $z = 0$ is x-simple, y-simple and z-simple. Thus we may attempt to compute the integral in three ways.

<u>1st way.</u> The orthogonal projection of T on the (y,z)-plane is the square $R = \{(y,z) : 0 \le y \le 1, 0 \le z \le 1\}$. Thus

$$\int\int\int_T 4xyz\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int\int_R \left(\int_{-\sqrt{1-y}}^{\sqrt{1-y}} 4xyz\, \mathrm{d}x\right) \mathrm{d}y\, \mathrm{d}z$$

$$= \int\int_R 2yz\left((\sqrt{1-y})^2 - (-\sqrt{1-y})^2\right) \mathrm{d}x\, \mathrm{d}y = \int\int_R 0\, \mathrm{d}x\, \mathrm{d}y = 0$$

<u>2nd way.</u> The orthogonal projection of T on the (x,z)-plane is the rectangle $S = \{(x,z) : -1 \le x \le 1, 0 \le z \le 1\}$. Thus

$$\int\int\int_T 4xyz\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int\int_S \left(\int_0^{1-x^2} 4xyz\, \mathrm{d}y\right) \mathrm{d}x\, \mathrm{d}z =$$

$$\int\int_S 2xz(1 - x^2)^2)\, \mathrm{d}x\, \mathrm{d}z = \int_0^1 2z\left(\int_{-1}^{1} x(1-x^2)^2\, \mathrm{d}x\right) \mathrm{d}z.$$

Since the function $x(1-x^2)^2$ is odd, we have $\int_{-1}^{1} x(1-x^2)^2 dx = 0$ (it is also easy to verify this directly). Thus

$$\int\int\int_T 4xyz\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_0^1 2z \cdot 0\, \mathrm{d}z = 0.$$

<u>3rd way</u>. The orthogonal projection of T on the (x,y)-plane is the region B bounded by the parabola $y = 1 - x^2$ and the x-axis. Thus

$$\int\int\int_T 4xyz\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int\int_B \left(\int_0^1 4xyz\, \mathrm{d}z\right) \mathrm{d}x\mathrm{d}y = \int\int_B 2xy\, \mathrm{d}x\, \mathrm{d}y =$$

$$= \int_{-1}^{1} \left(\int_0^{1-x^2} 2xy \, \mathrm{d}y \right) \mathrm{d}x = \int_{-1}^{1} x(1-x^2)^2 \, \mathrm{d}x = 0.$$

c) We have

$$\int_0^1 \int_{\sqrt{y}}^1 3e^{x^3} \, \mathrm{d}x \, \mathrm{d}y = \int\int_R 3e^{x^3} \, \mathrm{d}x \, \mathrm{d}y$$

where R is the region bounded by the lines $y = 0$, $x = 1$ and the parabola $y = x^2$. This region is vertically simple so

$$\int\int_R 3e^{x^3} \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_0^{x^2} 3e^{x^3} \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 3x^2 e^{x^3} \, \mathrm{d}x = e - 1.$$

Solution to Problem 3. a) The region R in the first quadrant, bounded by the curve $r = \sqrt{\pi + \theta}$, is radially simple so we use integration is polar coordinates:

$$\int\int_R 2\cos(x^2+y^2) \, \mathrm{d}x \, \mathrm{d}y = \int_0^{\pi/2} \int_0^{\sqrt{\pi+\theta}} 2\cos(r^2) r \, \mathrm{d}r \, \mathrm{d}\theta = \int_0^{\pi/2} \sin(\pi+\theta) \, \mathrm{d}\theta = -1$$

b) We use spherical coordinates. In spherical coordinates the part T of the ball $x^2 + y^2 + z^2 \leq 1$ in the first octant is given by $T = \{(\rho, \phi, \theta) : 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq \pi/2\}$. Thus

$$\int\int\int_T 4z\sqrt{1 + (x^2+y^2+z^2)^2} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 4\rho\cos\phi\sqrt{1+\rho^4}\rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi\cos\phi \left(\int_0^1 \sqrt{1+\rho^4} 4\rho^3 \, \mathrm{d}\rho \right) \mathrm{d}\theta \, \mathrm{d}\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi\cos\phi \left(\frac{2}{3}(2\sqrt{2}-1) \right) \mathrm{d}\theta \, \mathrm{d}\phi = \frac{2}{3}(2\sqrt{2}-1) \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi\cos\phi \, \mathrm{d}\theta \, \mathrm{d}\phi =$$

$$= \frac{2}{3}(2\sqrt{2}-1)\frac{\pi}{2} \int_0^{\pi/2} \sin\phi\cos\phi \, \mathrm{d}\phi = \frac{\pi}{3}(2\sqrt{2}-1)\frac{1}{2} = \frac{\pi}{6}(2\sqrt{2}-1).$$

Solution to Problem 4. a) The curve C has parametrization $x = t$, $y = 2t$, $t \in [0, 1]$. Thus

$$\int_C e^{4x^2-y^2} \mathrm{d}s = \int_0^1 e^{4t^2-(2t)^2} \sqrt{1^2+2^2} \, \mathrm{d}t = \sqrt{5}.$$

c) We know that

$$\int_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \int_C -y \, \mathrm{d}x + x \, \mathrm{d}y + (z - x^2 - y^2) \, \mathrm{d}z =$$

$$= \int_0^{\pi} (-\sin t(-\sin t) + \cos t \cos t + (t - \cos^2 t - \sin^2 t))\ dt = \int_0^{\pi} t\ dt = \frac{\pi^2}{2}.$$

Solution to Problem 5. The surface is given by the parametric equation

$$\mathbf{r}(u, v) =< \cos u, \sin u, z = u + v >,$$

where $0 \le u \le \pi$ and $0 \le v \le 1$. We have $\mathbf{r}_u =< -\sin u, \cos u, 1 >$ and $\mathbf{r}_v =< 0, 0, 1 >$. Thus $\mathbf{r}_u \times \mathbf{r}_v =< \cos u, \sin u, 0 >$ and $|\mathbf{r}_u \times \mathbf{r}_v| = 1$. The area of this surface is

$$\int_0^{\pi} \int_0^1 |\mathbf{r}_u \times \mathbf{r}_v|\ du\ dv = \int_0^{\pi} \int_0^1 1\ du\ dv = \pi.$$

Solution to Problem 6. The volume of the solid R between the paraboloids $z = x^2 + y^2$, $z = 4(x^2 + y^2)$ and the planes $z = 1$, $z = 4$ equals $V = \int\int\int_R 1\ dx\ dy\ dz$. Since the map

$$F(r, t, \theta) = (\frac{r}{t} \cos\theta, \frac{r}{t} \sin\theta, r^2)$$

takes the box $B = \{(r, t, \theta) : 1 \le r \le 2, 1 \le t \le 2, 0 \le \theta \le 2\pi\}$ bijectively onto R, we may use the change of variables formula

$$\int\int\int_R 1\ dx\ dy\ dz = \int\int\int_B |JF|\ dr\ dt\ d\theta,$$

where JF is the Jacobian of F. The Jacobian is the determinant of the matrix

$$\begin{pmatrix} \frac{1}{t}\cos\theta & \frac{-r}{t^2}\cos\theta & \frac{-r}{t}\sin\theta \\ \frac{1}{t}\sin\theta & \frac{-r}{t^2}\sin\theta & \frac{r}{t}\cos\theta \\ 2r & 0 & 0 \end{pmatrix}.$$

Expansion along the third row gives

$$JF = 2r(\frac{-r^2}{t^3}\cos^2\theta - \frac{r^2}{t^3}\sin^2\theta) = \frac{-2r^3}{t^3}.$$

Thus

$$V = \int\int\int_B \frac{2r^3}{t^3}\ dr\ dt\ d\theta = \int_0^{2\pi} \int_1^2 \int_1^2 \frac{2r^3}{t^3}\ dr\ dt\ d\theta = \int_0^{2\pi} \int_1^2 \frac{15}{2t^3}\ dt\ d\theta =$$

$$= \int_0^{2\pi} \frac{45}{16}\ d\theta = \frac{45\pi}{8}.$$