Some problems on divisibility

We have proved in class and homework the following results.

1. If a, b, c are integers such that $c|ab$ and $\gcd(c, a) = 1$ then $c|b$.

2. If a, b, c are integers such that $a|c$ and $b|c$ and $\gcd(a, b) = 1$ then $ab|c$.

3. If a, b, c are integers such that $\gcd(a, c) = 1 = \gcd(b, c)$ then $\gcd(ab, c) = 1$.

You should be able to prove these results. Also you should be able to prove the following:

4. If a, b are integers and $d = \gcd(a, b)$ then the numbers a/d, b/d are relatively prime.

5. If $\gcd(a, b) = 1$ then $\gcd(a, b^n) = 1$ for any natural number n.

6. If $\gcd(a, b) = 1$ then $\gcd(a^m, b^n) = 1$ for any natural numbers m, n.

7. If $\gcd(a^m, b^n) = 1$ for some natural numbers m, n then $\gcd(a, b) = 1$.

8. If n is a natural number and a, b are integers such that $a^n|b^n$ then $a|b$. Conclude that if r is a rational number such that r^n is an integer then r is also an integer.

9. Let a, b, c be non-zero integers. Consider the set $S = \{xa + yb + zc : x, y, z \in \mathbb{Z}\}$. Prove that S contains a positive integer d which divides each of a, b,c. Conclude that d is the largest common divisor of a, b, and c and that d is the smallest positive element of S.