Exam 2, Math 401
Tuesday, October 30

Problem 1. a) Define prime and irreducible elements in an integral domain R. (5 points)

b) Let I, J be ideals of a ring R. Define $I + J$ and IJ. (5 points)

c) Define $< a_1, ..., a_k >$, where $a_1, ..., a_k$ are elements of a ring R. Define Noetherian ring. (5 poinst)

d) State the First Isomorphism Theorem (5 points)

e) Define an Euclidean domain. Define unique factorization domain. (6 points)

Problem 2. a) Define an ideal in a ring R. Define a prime ideal. Define principal ideal. (7 points)

b) Let R be a commutative ring and let $a \in R$. Set $\text{ann}(a) = \{r \in R : ra = 0\}$ (this set is called the **annihilator** of a). Prove that $\text{ann}(a)$ is an ideal in R. (6 points)

c) Let $R = \mathbb{Z}/24$ and let $a = 20$. Find the ideal $\text{ann}(a)$ (it should be of the form $m\mathbb{Z}/24$ for some divisor m of 24). (6 points)

d) Let P be a prime ideal in a commutative ring R. Suppose that $a \in R$ but $a \notin P$. Prove that $\text{ann}(a) \subseteq P$. (6 points)

Problem 3. a) State the Division Algorithm for polynomials. Explain how does this result imply that polynomial rings over fields are Euclidean domains. (8 points)

b) Find a greatest common divisor of the polynomials $p = x^5 + x^4 + x^3 + x^2 + x + 1$ and $q = x^3 - 1$ in $\mathbb{Q}[x]$. (7 points)

c) Which of the polynomials $x^4 + 4$, $x^3 + x + 1$, $x^2 + 3$ in $\mathbb{F}_5[x]$ are irreducible? Justify your answer. Factor each of these polynomials into irreducible factors. (Here $\mathbb{F}_5$ is the field $\mathbb{Z}/5$). (10 points)

Problem 4. a) Let R be PID. Consider two elements $a, b \in R$. Since R is a PID, there is $d \in R$ such that $aR \cap bR = dR$. Prove that for any $c \in R$ we have $d|c$ iff $a|c$ and $b|c$. What would be appropriate name for d? (12 points)

b) Let $R = \mathbb{Z}[\sqrt{6}] = \{a + b\sqrt{6} : a, b \in \mathbb{Z}\}$. Consider the ideal $I =< 2, \sqrt{6} >$. Prove that I and $1 + I$ are different cosets of I in R. Prove that these are the only cosets. What can you say about R/I? (12 points)

****

****

The following problems are optional. You may earn extra points, but work on these problems only after you are done with the other problems

Problem 5. Let $R = \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$. Define $N(a + b\sqrt{-2}) = a^2 + 2b^2$ (so N is just the square of the absolute value of the complex number $a + b\sqrt{-2}$. Suppose that $0 \neq x = a + b\sqrt{-2}$ and $y = c + d\sqrt{-2}$ are elements of R. Prove that the complex number y/x can be expressed as $s + t\sqrt{-2}$ for some rational numbers s, t. Use N to prove that R is Euclidean. (10 points)

Problem 6. Let $R = \mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$ (so this ring is a subring of the Eisenstein integers).

a) Prove that $1, -1$ are the only invertible elements in R. (5 points)

b) Prove that 2, $1 + \sqrt{-3}$, $1 - \sqrt{-3}$ are irreducible in R. Conclude that R is not UFD (find 2 inequivalent factorizations of 4). (5 points)

c) Prove that the ideal $I =< 2, 1 + \sqrt{-3} >$ of R is not principal and that it is maximal. (5 points)