Exam 3, Math 401

Tuesday, November 27

Problem 1. In the group S_9 let $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 8 & 1 & 7 & 3 & 2 & 6 & 5 \end{pmatrix}$.

a) Write a as a product of disjoint cycles. (3 points)

b) Find the order of a. (2 points)

c) Compute $a(2,5,3,7,4)a^{-1}$. (3 points)

d) Write a as a product of transpositions. (3 points)

e) Is a even or odd? (2 points)

f) Is there an element of order 16 in S_9? Explain your answer. (3 points)

g) Let $b = a^2$. Write b as a product of disjoint cycles. List all elements of $< b >$. Is $< b >$ a normal subgroup? (4 points)

Problem 2. a) Prove that any 2 elements of order 3 in S_5 are conjugate. Is the same true for S_6? Hint: What can you say about elements of order 3 in S_5? (7 points)

b) Suppose that a subgroup H of S_6 contains $\sigma = (1,6)$ and $\tau = (2,3,4,5,6)$. Prove that $H = S_6$. Hint: What is $(1,i)(1,j)(1,i)$? (7 points)

Problem 3. a) Define a normal subgroup of a group G (7 points).

b) Let H and K be normal subgroups of a group G. Prove that if $a \in K$ and $b \in H$ then $aba^{-1}b^{-1} \in K \cap H$. Conclude that if $K \cap H = \{e\}$ then every element of K commutes with every element of H. (7 points)

c) Suppose that the set $N = \{a \in G : a^3 = e\}$ is a subgroup of a group G. Prove that it is a normal subgroup. (7 points)

Problem 4. a) State Lagrange's Theorem. (8 points)

b) The group S_4 can be considered as the group of all permutations of vertices $1, 2, 3, 4$ of a square (numbered counterclockwise). It contains as a subgroup the dihedral group D_8 of order 8 (which consists of those permutations which are isometries; thus $T = (1, 2, 3, 4)$ and $S = (2, 4)$). What is the index $[S_4 : D_8]$? Prove that D_8 is not normal in S_4? (7 points)

c) Let G be a finite group with a normal subgroup N and a subgroup H such that $\gcd(|H|, [G : N]) = 1$. Prove that $H \subseteq N$. Hint: Either use the Third Isomorphism Theorem or study the canonical homomorphism $G \longrightarrow G/N$. (7 points)

Problem 5. a) State the First Isomorphism Theorem for groups. (8points)

b) Let $G =< g >$ be a cyclic group of order n. For each integer m define a map $f_m : G \longrightarrow G$ by $f_m(a) = a^m$.

1. Prove that f_m is a homomorphism. (5 points)
2. Prove that f_m is an automorphism iff $\gcd(m, n) = 1$ (5 points)
3. Find the kernel and the image of f_8 when $n = 12$. (5 points)

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The following problems are optional. You may earn extra points, but work on these problems only after you are done with the other problems

Problem 6. a) Prove that if $n \geq 3$ then S_n has no normal subgroups of order 2.

b) Let $n \geq 5$. Prove that if N is a normal subgroup of S_n then $N = \{e\}$, $N = A_n$ or $N = S_n$. Hint: Note that $N \cap A_n$ is normal in A_n.

Problem 7. In Problem 4b) above list all elements of $D_8 \cap A_4$. Prove that the non-trivial elements of $D_8 \cap A_4$ are exactly the permutations of S_4 which are products of two disjoint transpositions. Conclude that $D_8 \cap A_4$ is normal in S_4. Consider the subset F of S_4 which consists of all permutations which take 1 to 1. Prove that F is a subgroup of S_4 and it is isomorphic to S_3. Prove that $F \cap (D_8 \cap A_4) = \{e\}$. Conclude that $S_4/(D_8 \cap A_4)$ is isomorphic to S_3.