**Solutions to Exam 3**

**Problem 1.** In the group $S_9$ let $a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 8 & 1 & 7 & 3 & 2 & 6 & 5 \end{pmatrix}$.

a) Write $a$ as a product of disjoint cycles. (3 points)

b) Find the order of $a$. (2 points)

c) Compute $a(2, 5, 3, 7, 4)a^{-1}$. (3 points)

d) Write $a$ as a product of transpositions. (3 points)

e) Is $a$ even or odd? (2 points)

f) Is there an element of order 16 in $S_9$? Explain your answer. (3 points)

g) Let $b = a^2$. Write $b$ as a product of disjoint cycles. List all elements of $< b >$. Is $< b >$ a normal subgroup? (4 points)

**Solution:** a) $a = (1, 4)(2, 9, 5, 7)(3, 8, 6)$.

b) The order of $a$ is equal to $\mathrm{lcm}(2, 4, 3) = 12$.

c) $a(2, 5, 3, 7, 4)a^{-1} = (a(2), a(5), a(3), a(7), a(4)) = (9, 7, 8, 2, 1)$.

d) Since $(2, 9, 5, 7) = (2, 7)(2, 5)(2, 9)$ and $(3, 8, 6) = (3, 6)(3, 8)$, we have

$$a = (1, 4)(2, 7)(2, 5)(2, 9)(3, 6)(3, 8).$$

e) Since $a$ is a product of 6 transpositions, it is even.

f) Recall that the order of a permutation $\tau$ is the least common multiple of the lengths of the cycles in the cycle decomposition of $\tau$. Note that if a least common multiple of some integers is 16 then one of the integers must be equal to 16 (this is true for any prime power in place of 16). Thus a permutation of order 16 must have at least one cycle of length 16 but $S_9$ does not have cycles of length larger than 9. Thus there is non element of order 16 in $S_9$.

g) We have $b = a^2 = (1, 4)^2(2, 9, 5, 7)^2(3, 8, 6)^2 = (2, 5)(9, 7)(3, 6, 8)$. Now $b^2 = (3, 8, 6)$, $b^3 = (2, 5)(9, 7)$, $b^4 = (3, 6, 8)$, $b^5 = (2, 5)(9, 7)(3, 8, 6)$ and $b^6 = e$ is the identity. Thus $< b >= \{e, b, b^2, b^3, b^4, b^5\}$. It is not a normal subgroup, since $(3, 8, 6) = b^2 \in< b >$ but $(3, 1)(3, 8, 6)(3, 1)^{-1} = (1, 8, 6) \notin< b >$.

**Problem 2.** a) Prove that any 2 elements of order 3 in $S_5$ are conjugate. Is the same true for $S_6$? Hint: What can you say about elements of order 3 in $S_5$? (7 points)

b) Suppose that a subgroup $H$ of $S_6$ contains $\sigma = (1, 6)$ and $\tau = (2, 3, 4, 5, 6)$. Prove that $H = S_6$. Hint: What is $(1, i)(1, j)(1, i)$? (7 points)

**Solution:** a) In general, a permutation of prime order $p$ is a product of disjoint $p-$cycles. Thus an element of order 3 in $S_5$ is a product of disjoint $3-$cycles. Since $6 > 5$, only one three cycle can be present, so an element of order 3 in $S_5$ is a $3-$cycle. We know that any two $3-$cycles are conjugate, which proves the claim.

In $S_6$, $(1, 2, 3)$ and $(1, 2, 3)(4, 5, 6)$ have both order 3 but they are not conjugate since they have different type of cycle decomposition.

b) Note that $\tau\sigma\tau^{-1} = (1, 2) \in H$, $\tau(1, 2)\tau^{-1} = (1, 3) \in H$, $\tau(1, 3)\tau^{-1} = (1, 4) \in H$, $\tau(1, 4)\tau^{-1} = (1, 5) \in H$. Note now that $(1, i)(1, j)(1, i) = (i, j) \in H$, for any $1 < i < j$. Thus $H$ contains all transpositions. Since every element in $S_6$ is a product of transpositions, in belongs to $H$, so $S_6 = H$.

**Problem 3.** a) Define a normal subgroup of a group $G$ (7 points).

b) Let $H$ and $K$ be normal subgroups of a group $G$. Prove that if $a \in K$ and $b \in H$ then $aba^{-1}b^{-1} \in K \cap H$. Conclude that if $K \cap H = \{e\}$ then every element of $K$ commutes with every element of $H$. (7 points)

c) Suppose that the set $N = \{a \in G : a^3 = e\}$ is a subgroup of a group $G$. Prove that it is a normal subgroup. (7 points)

**Solution:** a) A subgroup $H$ of $G$ is called normal (notation: $H \lhd G$) if $ghg^{-1} \in H$ for every $g \in G$ and every $h \in H$. The following properties are equivalent:

1. $H \lhd G$;
2. $c_g(H) \subseteq H$ for every $g \in G$, where $c_g$ is the conjugation by $g$;
3. $c_g(H) = H$ for every $g \in G$;

4. the sets of left and right cosets of $H$ in $G$ coincide;
5. $aH = Ha$ for every $a \in G$;
6. $H = \ker f$ for some homomorphism $f : G \longrightarrow D$.

b) Since $K \lhd G$, $a \in K$, and $b \in G$, we have $a^{-1} \in K$, and $ba^{-1}b^{-1} \in K$, and $aba^{-1}b^{-1} \in K$. Similarly, since $H \lhd G$, $b \in H$, and $a \in G$, we have $aba^{-1} \in H$ and $aba^{-1}b^{-1} \in H$. Thus $aba^{-1}b^{-1} \in K \cap H$.

If $K \cap H = \{e\}$, then for every $a \in K$ and every element $b \in H$ we have $aba^{-1}b^{-1} = e$, i.e. $ab = ba$.

c) Let $a \in N$ and $g \in G$. Since conjugation by $g$ is a homomorphisms, we have

$$(gag^{-1})^3 = ga^3g^{-1} = geg^{-1} = e$$

so $gag^{-1} \in N$. This proves that $N$ is normal in $G$.

**Problem 4.** a) State Lagrange's Theorem. (8 points)

b) The group $S_4$ can be considered as the group of all permutations of vertices $1, 2, 3, 4$ of a square (numbered counterclockwise). It contains as a subgroup the dihedral group $D_8$ of order 8 (which consists of those permutations which are isometries; thus $T = (1, 2, 3, 4)$ and $S = (2, 4)$). What is the index $[S_4 : D_8]$? Prove that $D_8$ is not normal in $S_4$? (7 points)

c) Let $G$ be a finite group with a normal subgroup $N$ and a subgroup $H$ such that $\gcd(|H|, [G : N]) = 1$. Prove that $H \subseteq N$. Hint: Either use the Third Isomorphism Theorem or study the canonical homomorphism $G \longrightarrow G/N$. (7 points)

**Solution:** a) **Lagrange's Theorem.** Let $G$ be a finite group with a subgroup $H$. Then $|G| = |H|[G : H]$.

b) Since $|S_4| = 24$ and $D_8 = 8$, we have $[S_4 : D_8] = 24/8 = 3$ by Lagrange's Theorem.

Note that $(2, 4) \in D_8$ but $(1, 4)(2, 4)(1, 4)^{-1} = (2, 1) \notin D_8$. Thus $D_8$ is not normal in $S_4$.

**Another argument:** Suppose that $D_8$ is normal. Consider the canonical homomorphism $\pi : S_4 \longrightarrow S_4/D_8 = K$. Note that $|K| = 3$. If $\tau$ is a transposition, then the order of $\pi(\tau)$ divides both the order of $\tau$ and the order of $K$. Since the former is 2 and the latter is 3, we see that the order of $\pi(\tau)$ is 1, i.e. $\tau \in \ker \pi = D_8$. This means that $D_8$ contains all transpositions and hence is equal to $S_4$, a contradiction. Thus $D_8$ is not normal in $S_4$.

c) By the Third Isomorphism Theorem, the groups $HN/N$ and $H/(H \cap N)$ are isomorphic, hence have the same order, call it $m$. Since $HN/N$ is a subgroup of $G/N$, its order divides $|G/N| = [G : N]$, i.e. $m|[G : N]$. On the other hand, $m = |H/(H \cap N)| = [H : H \cap N]$ divides $|H|$. Thus $m$ divides both $|H|$ and $[G : N]$. Since $\gcd(|H|, [G : N]) = 1$, me must have $m = 1$. this means that $H = H \cap N$, i.e. $H \subseteq N$.

**Another argument:** Consider the canonical homomorphism $\pi : G \longrightarrow G/N = K$. The image $\pi(H)$ is a subgroup of $K$, so $|\pi(H)|$ divides $|K| = [G : N]$. On the other hand, $\pi$ maps $H$ onto $\pi(H)$, so $\pi(H)$ is isomorphic to $H/(\ker \pi \cap H)$ by the First Isomorphism Theorem. By Lagrange's Theorem, $|\pi(H)|$ divides $|H|$. Thus $|\pi(H)|$ divides both $|H|$ and $[G : N]$. Since $\gcd(|H|, [G : N]) = 1$, we must have $|\pi(H)| = 1$, i.e. $H \subseteq \ker \pi = N$.

**Problem 5.** a) State the First Isomorphism Theorem for groups. (8points)

b) Let $G =< g >$ be a cyclic group of order $n$. For each integer $m$ define a map $f_m : G \longrightarrow G$ by $f_m(a) = a^m$.

1. Prove that $f_m$ is a homomorphism. (5 points)
2. Prove that $f_m$ is an automorphism iff $\gcd(m, n) = 1$ (5 points)
3. Find the kernel and the image of $f_8$ when $n = 12$. (5 points)

**Solution:** a) **First Isomorphism Theorem.** Let $f : G \longrightarrow H$ be a homomorphism with kernel $K = \ker f$. The map $g : G/K \longrightarrow f(K)$ defined by $g(aK) = f(a)$ is an isomorphism. In particular, if $f$ is surjective, then $G/K$ and $H$ are naturally isomorphic.

b) Since $G$ is abelian, we have $f_m(ab) = (ab)^m = a^m b^m = f_m(a)f_m(b)$. This shows that $f_m$ is a homomorphism.

Suppose that $a \in G$ belongs to the kernel of $f$. If $k$ is the order of $a$ then $k||G| = n$. But also $e = f_m(a) = a^m$, so $k|m$. Thus, if $\gcd(m,n) = 1$, then the kernel of $f_m$ is trivial, i.e. $f_m$ is injective. An injective function from a finite set to itself is also surjective, so $f_m$ is an automorphism. Conversely, if $d > 1$ is a divisor of both $n$ and $m$ then $g^{n/d} \neq e$ but $f_m(g^{n/d}) = g^{nm/d} = (g^n)^{m/d} = e$, so $f_m$ is not injective. This proves 2.

Note that for $n = 12$ we have $f_8(g^i) = e$ iff $g^{8i} = e$, iff $12|8i$, iff $3|i$. Thus $\ker f_8 = \{e, g^3, g^6, g^9\} =< g^3 >$. The image of $f_8$ is $< g^8 >= \{g^8, g^4, e\}$.

**Remark.** In general, let $d = \gcd(m,n)$. Note that $e = f_m(g^i) = g^{mi}$ iff $n|mi$, iff $n/d$ divides $i$ (since $n/d$ and $m/d$ are relatively prime). Thus the kernel of $f_m$ is $< g^{n/d} >$, so it has $d$ elements. Note that $d = sm + tn$ for some integers $s, t$ so $f_m(g^s) = g^{sm} = g^{em}g^{tn} = g^{sm+tn} = g^d$ (since $g^{tn} = e$). Thus $< g^d >$ is contained in the image of $f_m$. Note that $g^d$ has order $n/d$, so $| < g^d > | = n/d$. By Lagrange's Theorem and the First Isomorphism Theorem, the image of $f_m$ has $|G|/|\ker f_m| = n/d$ elements. It follows that $f_m(G) =< g^d >$.

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The following problems are optional. You may earn extra points, but work on these problems only after you are done with the other problems

**Problem 6.** a) Prove that if $n \geq 3$ then $S_n$ has no normal subgroups of order 2.

b) Let $n \geq 5$. Prove that if $N$ is a normal subgroup of $S_n$ then $N = \{e\}$, $N = A_n$ or $N = S_n$. Hint: Note that $N \cap A_n$ is normal in $A_n$.

**Solution:** a) Suppose that $M = \{e, \tau\}$ is a subgroup of order 2. Since $\tau$ is not trivial, $\tau(a) = b \neq a$ for some $a \in \{1, 2, ..., n\}$. Since $n \geq 3$, there is $c \in \{1, 2, ..., n\}$ such that $c \neq a$ and $c \neq b$. Then $[(b,c)\tau(b,c)^{-1}](a) = c \neq \tau(a)$, so $(b,c)\tau(b,c)^{-1} \neq \tau$ and $(b,c)\tau(b,c)^{-1} \neq e$. Thus $(b,c)\tau(b,c)^{-1} \notin M$, which shows that $M$ is not normal in $S_n$.

b) Suppose that $n \geq 5$. Then $A_n$ is a simple group. Suppose that $N$ is a non-trivial proper normal subgroup of $S_n$. Then $N \cap A_n$ is normal in $A_n$. Thus either $N \cap A_n = \{e\}$ or $N \cap A_n = A_n$. In the former case, the canonical homomorphism

$S_n \longrightarrow S_n/A_n$ is injective on $N$ and since $S_n/A_n$ has order 2, also $N$ has order 2. This is not possible by a). In the latter case, $A_n \subseteq N$, so $N/A_n$ is a proper subgroup of $S_n/A_n$. Since $S_n/A_n$ has only two elements, $N/A_n$ is trivial, i.e. $N = A_n$. This proves that $A_n$ is the only non-trivial proper normal subgroup of $S_n$ for $n \geq 5$.

**Problem 7.** In Problem 4b) above list all elements of $D_8 \cap A_4$. Prove that the non-trivial elements of $D_8 \cap A_4$ are exactly the permutations of $S_4$ which are products of two disjoint transpositions. Conclude that $D_8 \cap A_4$ is normal in $S_4$. Consider the subset $F$ of $S_4$ which consists of all permutations which take 1 to 1. Prove that $F$ is a subgroup of $S_4$ and it is isomorphic to $S_3$. Prove that $F \cap (D_8 \cap A_4) = \{e\}$. Conclude that $S_4/(D_8 \cap A_4)$ is isomorphic to $S_3$.

**Solution:** Note that

$$D_8 = \{e, (1,2,3,4) = T, (1,3)(2,4) = T^2, (1,4,3,2) = T^3,$$

$$(2,4) = S, (1,4)(2,3) = ST, (1,3) = ST^2, (1,2)(3,4) = ST^3\}.$$

Thus,

$$D_8 \cap A_4 = \{e, (1,3)(2,4), (1,4)(2,3), 1,2)(3,4)\}.$$

It is clear now that the non-trivial elements of $D_8 \cap A_4$ are exactly the permutations of $S_4$ which are products of two disjoint transpositions. Note that a conjugate of a product of two disjoint transpositions is again a product of two disjoint transpositions. This shows that $D_8 \cap A_4$ is normal in $S_4$. Suppose that $\sigma, \tau \in F$. Then $\sigma(1) = 1 = \tau(1)$. Thus $(\sigma\tau)(1) = \sigma(\tau(1)) = \sigma(1) = 1$, so $\sigma\tau \in F$. Also, $\sigma^{-1}(1) = 1$, so $\sigma^{-1} \in F$. We see that $F$ is a subgroup of $A_4$. Elements of $F$ can be identified with permutations of $\{2,3,4\}$, which in turn form a group isomorphic to $S_3$. Clearly no non-trivial element of $D_8 \cap A_4$ belongs to $F$. It follows that the canonical homomorphism $\phi : S_4 \longrightarrow S_4/(D_8 \cap A_4)$ is injective on $F$. Since $F$ and $S_4/(D_8 \cap A_4)$ have both 6 elements, $\phi$ gives an isomorphism of $F$ and $S_4/(D_8 \cap A_4)$. This proves that $S_4/(D_8 \cap A_4)$ is isomorphic to $S_3$.