Solution to problem 2.3. Since the set of all 2×2 matrices is a ring, a subset S with the same multiplication and addition is a ring iff $a - b$ and $a \cdot b$ are in S for any $a, b \in S$.

a) The product of two symmetric matrices need not be symmetric:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 3 & 5 \end{pmatrix}.$$

Thus symmetric matrices do not form a ring.

b) The product of two skew-symmetric matrices need not be skew-symmetric:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus skew-symmetric matrices do not form a ring.

c) The difference and the product of any two upper-traingular matrices is upper-triangular:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} - \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} a-x & b-y \\ 0 & c-z \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} ax & ay+bz \\ 0 & cz \end{pmatrix}.$$

Thus upper-traingular matrices form a ring. This ring is not commutative:

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since the identity matrix is upper-triangular, this ring is unital. It is not a division ring since the matrices above are not invertible (they are zero-divisors).

d) Clearly the difference of any two strictly upper-triangular matrices is strictly upper-triangular and the product of any two strictly upper-triangular matrices is the zero matrix (which is strictly upper-triangular), so the strictly upper-triangular matrices form a ring. The multiplication in this ring is trivial (the product of any two elements is zero) so this ring is commutative but does not have a unit and is not a division ring.

e) Since

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} - \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} a-x & b-y \\ -(b-y) & a-x \end{pmatrix} \text{ and } \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} ax-by & ay+bx \\ -bx-ay & -by+ax \end{pmatrix},$$

this set is a ring. Furthermore,

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} ax - by & ay + bx \\ -bx - ay & -by + ax \end{pmatrix} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

so this ring is commutative. It contains the identity matrix, so it is unital. Furthermore, if one of a, b is not zero then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} \frac{a}{a^2+b^2} & \frac{-b}{a^2+b^2} \\ \frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so every non-zero elemet has an inverse. Thus this ring is a field.

Solution to problem 2.9. We need to verify all the axioms.

A1.

$$(r_1, s_1) + (r_2, s_2)] + (r_3, s_3) = (r_1 + r_2, s_1 + s_2) + (r_3, s_3) = ((r_1 + r_2) + r_3, (s_1 + s_2) + s_3)$$

$$= (r_1 + (r_2 + r_3), s_1 + (s_2 + s_3)) = (r_1, s_1) + (r_2 + r_3, s_2 + s_3) = (r_1, s_1) + [(r_2, s_2) + (r_3, s_3)].$$

A2. We have $(r, s) + (0, 0) = (r + 0, s + 0) = (r, s) = (0, 0) + (r, s)$ so $(0, 0)$ is the zero of $R \times S$.

A3. Since $(r, s) + (-r, -s) = (r + (-r), s + (-s)) = (0, 0)$, every element has its negative.

A4. $(r, s) + (r_1, s_1) = (r + r_1, s + s_1) = (r_1 + r, s_1 + s) = (r_1, s_1) + (r, s)$.

M1. $[(r_1, s_1) \cdot (r_2, s_2)] \cdot (r_3, s_3) = (r_1 \cdot r_2, s_1 \cdot s_2) \cdot (r_3, s_3) = ((r_1 \cdot r_2) \cdot r_3, (s_1 \cdot s_2) \cdot s_3) = (r_1 \cdot (r_2 \cdot r_3), s_1 \cdot (s_2 \cdot s_3)) = (r_1, s_1) \cdot (r_2 \cdot r_3, s_2 \cdot s_3) = (r_1, s_1) \cdot [(r_2, s_2) \cdot (r_3, s_3)]$.

D.

$$[(r_1, s_1) + (r_2, s_2)] \cdot (r_3, s_3) = (r_1 + r_2, s_1 + s_2) \cdot (r_3, s_3) = ((r_1 + r_2) \cdot r_3, (s_1 + s_2) \cdot s_3)$$

$$= (r_1 \cdot r_3 + r_2 \cdot r_3, s_1 \cdot s_3 + s_2 \cdot s_3) = (r_1 \cdot r_3, s_1 \cdot s_3) + (r_2 \cdot r_3, s_2 \cdot s_3) = (r_1, s_1) \cdot (r_3, s_3) + (r_2, s_2) \cdot (r_3, s_3)$$

and similarly

$$(r_3, s_3) \cdot [(r_1, s_1) + (r_2, s_2)] = (r_3, s_3) \cdot (r_1, s_1) + (r_3, s_3) \cdot (r_2, s_2)$$

.

Now $R \times S$ is commutative iff $(r,s)(r_1,s_1) = (r_1,s_1)(r,s)$ for any $r, r_1 \in R$ and $s, s_1 \in S$. This means that $rr_1 = r_1 r$ and $ss_1 = s_1 s$ for any $r, r_1 \in R$ and $s, s_1 \in S$ so indeed $R \times S$ is commutative iff both R and S are commutative. If 1_R, 1_S are the identities of R and S respectively, that

$$(r,s)(1_R, 1_S) = (r \cdot 1_R, s \cdot 1_S) = (r,s) = (1_R, 1_S)(r,s)$$

so $(1_R, 1_S)$ is the identity of $R \times S$. Conversely, if (x,y) is the identity of $R \times S$ then

$$(r,s)(x,y) = (r,s) = (x,y)(r,s)$$

for any $r \in R$ and $s \in S$. In other words,

$$rx = r = xr \text{ and } sy = s = ys$$

for any $r \in R$ and $s \in S$, so x is the identity of R and y is the identity of S.

Since $(r,0)(0,s) = (0,0)$, $R \times S$ has zero divisors so it cannot be a filed.