Solution to Problem 2.6: We use induction on n. For $n = 1$ the result is trivial. Suppose that for some n we have

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + ... + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n.$$

Then

$$(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n,$$

and

$$x(x+y)^n = \binom{n}{0}x^{n+1} + \binom{n}{1}x^n y + ... + \binom{n}{n-1}x^2y^{n-1} + \binom{n}{n}xy^n.$$

Since x and y commute, also y and $x+y$ commute and therefore

$$y(x+y)^n = (x+y)^n y = \binom{n}{0}x^n y + \binom{n}{1}x^{n-1}y^2 + ... + \binom{n}{n-1}xy^n + \binom{n}{n}y^{n+1}.$$

Thus,

$$(x+y)^{n+1} = \binom{n}{0}x^{n+1} + (\binom{n}{0} + \binom{n}{1})x^n y + (\binom{n}{1} + \binom{n}{2})x^{n-1}y^2 + ... + (\binom{n}{n-1} + \binom{n}{n})xy^n + \binom{n}{n}y^{n+1}.$$

Note that $\binom{n}{0} = \binom{n+1}{0}$, $\binom{n}{n} = \binom{n+1}{n+1}$ and $\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$ for $i = 1, 2, ..., n-1$. It follows that

$$(x+y)^{n+1} = \binom{n+1}{0}x^{n+1} + \binom{n+1}{1}x^n y + ... + \binom{n+1}{n}xy^n + \binom{n+1}{n+1}y^{n+1}.$$

This proves that the result holds for $n+1$. By induction, it holds for all natural numbers $n > 0$.