Homework
due on Tuesday, October 9

Read section 2.2.3 of Cameron's book.

Problem 1. Let $I = n\mathbb{Z}$ and $J = m\mathbb{Z}$ be ideals of $\mathbb{Z}$.

a) Prove that $I + J = \gcd(m, n)\mathbb{Z}$.

b) Prove that $I \cap J = [m, n]\mathbb{Z}$, where $[m, n]$ is the least common multiple of m and n.

c) Prove that $IJ = (mn)\mathbb{Z}$.

d) Prove that $I \subseteq J$ iff $m|n$.

Problem 2. Let R be a ring. Two ideals I, J of R are called **coprime** if $I + J = R$. Suppose that I and J are coprime.

a) Prove that for any $r, t \in R$ there is $s \in R$ such that $s + I = r + I$ and $s + J = t + J$. **Hint:** Write $r = i + j$, $t = i_1 + j_1$ for some $i, i_1 \in I$ and $j, j_1 \in J$ and consider $s = j + i_1$.

b) Let $f_I : R \longrightarrow R/I$ and $f_J : R \longrightarrow R/J$ be the cannonical homomorphisms. Define $f : R \longrightarrow (R/I) \oplus (R/J)$ by $f(r) = (f_I(r), f_J(r))$. Use a) to prove that f is a surjective ring homomorphism. What is $\ker f$?

c) Prove that $R/(I \cap J)$ is isomorphic to $(R/I) \oplus (R/J)$. Conclude that $\mathbb{Z}/(mn)\mathbb{Z}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ for any relatively prime integers m, n (compare this to the Chinese remainder theorem and the map r of Lemma 1.6.3 in Lauritzen's book..

d) Let R be unital and commutative. Prove that $I \cap J = IJ$. (**Hint:** Write $1 = i + j$ for some $i \in I$, $j \in J$ and use the fact that $x = 1 \cdot x$ for any x.) Conclude that $[m, n] = mn$ for relatively prime positive integers m, n.