Homework

due on Monday, October 15

Read sections 3.1 and 3.3 of Lauritzen's book. Solve the following problems:

Problem 1. An element a of a ring R is called **nilpotent** if $a^m = 0$ for some $m > 0$.

a) Prove that in a commutative ring R the set N of all nilpotent elements of R is an ideal. This ideal is called the **nilradical** of R. Prove that 0 is the only nilpotent element of R/N.

b) Let R be a commutative ring and let $a_1, ..., a_n \in R$ be nilpotent. Set I for the ideal $< a_1, ..., a_n >$ generated by $a_1, ..., a_n$. Prove that there is a positive integer N such that $x_1 x_2 ... x_N = 0$ for any $x_1, ..., x_N$ in I (i.e. that $I^N = 0$).

c) Prove that the set of all nilpotent elements in the ring $M_2(\mathbb{R})$ is not an ideal.

d) Prove that if p is a prime and $m > 0$ then every element of $\mathbb{Z}/p^m\mathbb{Z}$ is either nilpotent or invertible.

e) Find the nilradical of $\mathbb{Z}/36\mathbb{Z}$ (by correspondence theorem, it is equal to $n\mathbb{Z}/36\mathbb{Z}$ for some n).

Problem 2. Let R be a commutative ring. For an ideal I of R define

$$\sqrt{I} = \{x \in R : x^n \in I \text{ for some } n > 0\}.$$

a) Prove that $\sqrt{I}$ is an ideal. It is called the **radical** of I.

b) Prove that $\sqrt{\{0\}}$ is the nilradical of R.

c) Consider a surjective homomorphism $f : R \longrightarrow S$. Prove that in the correspondence theorem the nilradical of S corresponds to $\sqrt{\ker f}$.

d) Prove that $R/\sqrt{I}$ has trivial nilradical.

Problem 3. Let R be a commutative ring. Let $I =< a_1, ..., a_n >$, $J =< b_1, ..., b_m >$. Prove that IJ is generated by the mn elements $a_i b_j$, $i = 1, 2, ..., n$, $j = 1, 2, ..., m$.