Problem 1. Prove that $a \equiv b \pmod{c}$ if and only if a and b give the same remainders upon division by c.

Solution: Let r_a, r_b be the remainders of a, b upon division by c respectively. Thus $a \equiv r_a \pmod{c}$ and $b \equiv r_b \pmod{c}$. It follows that $a \equiv b \pmod{c}$ iff $r_a \equiv r_b \pmod{c}$, i.e. iff $c|(r_a - r_b)$. Note that $0 \leq r_a, r_b < c$, so $|r_a - r_b| < c$. Thus the only way for $c|(r_a - r_b)$ to hold is to have $r_a = r_b$.

Problem 2. a) Find the remainder upon division of 2^{85} by 341.

b) Find smallest $a > 0$ such that $2^a \equiv 1 \pmod{341}$.

Solution: a) Use succesive squarings. We have $85 = 2^6 + 2^4 + 2^2 + 2^0$.

$$2^{2^0} \equiv 2 \pmod{341},$$

$$2^{2^1} \equiv 4 \pmod{341},$$

$$2^{2^2} \equiv 16 \pmod{341},$$

$$2^{2^3} \equiv 16^2 \equiv 256 \equiv -85 \pmod{341},$$

$$2^{2^4} \equiv (-85)^2 \equiv 64 \pmod{341},$$

$$2^{2^5} \equiv 64^2 \equiv 4 \pmod{341},$$

$$2^{2^6} \equiv 4^2 = 16 \pmod{341},$$

Thus

$$2^{85} = 2^{2^6} \cdot 2^{2^4} \cdot 2^{2^2} \cdot 2^{2^0} \equiv 16 \cdot 64 \cdot 16 \cdot 2 = 2^{2^3} \cdot 2^7 \equiv (-85) \cdot 4 \cdot 2^5 \equiv (-340) \cdot 2^5 \equiv 2^5 \pmod{341}$$

b) Note that from a) we have $2^8 \equiv -85 \pmod{341}$. Since $4 \cdot 85 = 340$, we have

$$2^{10} = 2^8 \cdot 4 \equiv (-85) \cdot 4 = -340 \equiv 1 \pmod{341}.$$

Since $2^8 < 341$ and $2^9 = 512 \not\equiv 1 \pmod{341}$, $a = 10$ is the smallest positive integer such that $2^a \equiv 1 \pmod{341}$.