Homework

due on Wednesday, October 16

Read section 2.3.1 in Cameron's book and section 3.1-3.3 in Lauritzen's book. Solve the following problems:

Problem 1. Let $R = \mathbb{Z}[\sqrt{2}]$. Let p be a prime such that 2 is a quadratic residue modulo p so that $k^2 \equiv 2 \pmod{p}$ for some integer k. Let $I = pR = \{a + b\sqrt{2} : p|a, p|b\}$.

a) Prove that neither $k + \sqrt{2}$ nor $k - \sqrt{2}$ belong to I.

b) Use a) to prove that I is not a prime ideal.

Problem 2. Let $R = \mathbb{Z}[i]$ be the ring of Gaussian integers. Let p be a prime number. Prove that pR is a maximal ideal iff -1 is a quadratic non-residue modulo p (i.e. iff $p \equiv 3 \pmod{4}$). (Follow the example we discussed for $\mathbb{Z}[\sqrt{2}]$. If you find it easier, prove that $2R$ is not prime and $7R$ is prime. But the general case is not much different.)