Homework

due on Friday, October 19

Read sections 2.3.1-2.3.3 in Cameron's book and sections 3.5-3.5.2 in Lauritzen's book. Solve the following problems:

Problem 1. We proved in class the following result

Theorem 1. *Let R be a commutative unital ring and let $a \in R$ be an element which is not a zero divisors (so the sequence $a, a^2, a^3, ...$ does not contain* 0*). The set of ideals of R which are disjoint with the set $\{a, a^2, a^3, ...\}$ contains maximal elements, (i.e. ideals which are not contained in any larger ideal of this set) and any such ideal is prime.*

Use this theorem to prove that in a commutative unital ring the intersection of all prime ideals is equal to the nilradical (see problem 1 of homework 19 for definition, Problem 2 b) from homework 20 can be useful).

Problem 2. Let R be an integral domain. Suppose that $0 \neq a \in R$ is such that aR is a prime ideal. Prove that a is irreducible.

Remark. The ring R is not considered a prime ideal, i.e. prime ideals are proper (I might have forgotten to add this in the definition).

Problem 3. Consider the ring $R = \mathbb{Z}[\sqrt{n}] = \{a + b\sqrt{n} : a, b \in \mathbb{Z}\}$, where n is an integer which is not a square (so $\sqrt{n}$ is not rational).

a) Define a map $f : R \longrightarrow R$ by $f(a + b\sqrt{n}) = a - b\sqrt{n}$. Prove that f is an isomorphism.

b) Consider the map $N : R \longrightarrow \mathbb{Z}$ defined by $N(a + b\sqrt{n}) = a^2 - nb^2$. Prove that $N(xy) = N(x)N(y)$ for all $x, y \in R$. Prove that $x \in R$ is invertible iff $N(x) = \pm 1$. Prove that if $|N(x)|$ is a prime number then x is irreducible.

c) Prove that $4 + i$ is irreducible in $\mathbb{Z}[i]$. Prove that the only invertible elements of $\mathbb{Z}[i]$ are $1, -1, i, -i$.

d) Prove that $\mathbb{Z}[\sqrt{2}]$ has infinitely many invertible elements. **Hint:** Consider $1+\sqrt{2}$. Note that product of invertible elements is invertible.