Homework

due on Wednesday, October 24

Read sections 2.2.4, 2.3.4 in Cameron's book and sections 4.1, 4.2 in Lauritzen's book. Solve the following problems:

Problem 1. Find a greatest common divisor $d(x)$ of the polynomials $p(x) = x^3 + 4x^2 + x - 6$ and $q(x) = x^5 - 6x + 5$ in the ring $\mathbb{Q}[x]$ and find $a(x), b(x) \in \mathbb{Q}[x]$ such that $d(x) = a(x)p(x) + b(x)q(x)$.

Problem 2. Let I be an ideal of the ring R. Define $I[x]$ as the subset of $R[x]$ which consists of all the polynomials in $R[x]$ whose all coefficients belong to I. Prove that $I[x]$ is an ideal of $R[x]$ and that $R[x]/I[x]$ is naturally isomorphic to the polynomial ring $(R/I)[x]$.

Problem 3. Let $R = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. Consider the homomorphism $f : \mathbb{Z}[x] \longrightarrow R$ such that $f(m) = m$ for $m \in \mathbb{Z}$ and $f(x) = \omega$ (there is unique such homomorphism by the result form class).

a) Prove that $\ker f = (x^2 + x + 1)\mathbb{Z}[x]$ and conclude that R is naturally isomorphic to the ring $\mathbb{Z}[x]/(x^2 + x + 1)\mathbb{Z}[x]$.

b) Prove that $x^2 + x + 1$ is a prime element in $\mathbb{Z}[x]$.

c) Prove that the ideal $M =< 2, x^2 + x + 1 >$ is not principal.

d) Prove that $\mathbb{Z}[x]/M$ and $R/2R$ are isomorphic. Conclude that M is maximal.