Homework
due on Friday, November 2

Problem 1. Let R be an integral domain and let $a, b \in R$. An element $m \in R$ is called a **least common multiple** of a and b if

1. $a|m$ and $b|m$
2. if $c \in R$ and $a|c$ and $b|c$ then $m|c$.

a) Prove that if m_1 and m_2 are least common multiples of a and b then m_1 and m_2 are associated (so, a least common multiple, if exists, is unique up to an invertible factor).

b) Let R be a UFD. We proved that $\gcd(a, b)$ exists for any $a, b \in R$. Let a, b be nonzero elements of R. Prove that $a/\gcd(a, b)$ and $b/\gcd(a, b)$ are relatively prime.

c) Suppose that R is a UFD. Let a, b be nonzero elements of R. Prove that $ab/\gcd(a, b)$ is a least common multiple of a and b. Thus least common multiples exist in any UFD.

Problem 2. Ler R be an integral domain.

a) Let $f, g \in R[x]$ be such that $fg = cx^n$ for some n and some $c \in R$, $c \neq 0$. Prove that there exist elements $a, b \in R$ and $m \leq n$ such that $f = ax^m$ and $g = bx^{n-m}$ and $ab = c$.

b) Suppose that $f = f_0 + f_1x + ... + f_nx^n \in R[x]$. Suppose that there is a prime ideal P of R such that $f_n \notin P$, $f_0, ..., f_{n-1} \in P$ and $f_0 \notin P^2$. Prove that if $f = gh$ for some $g, h \in R[x]$ then one of g, h is constant. Conclude that if in addition f is monic then it is irreducible in $R[x]$. This result is known as **Eisenstein criterion**. Hint: Assume that $f = gh$ and both g, h have positive degree. Pass to the ring $(R/P)[x]$ and apply a) to show that constant terms of g and h belong to P. Derive contradiction.

c) Prove that the polynomial $2x^{10} + 21x^8 - 35x^2 + 14$ is irreducible in $\mathbb{Z}[x]$. Hint: Apply Eisenstein criterion with appropriate prime ideal P.