Homework

due on Monday, November 5

Read sections 3.4 in Lauritzen's book and sections 2.4, 7.2.3, 7.2.4 in Cameron's book. Solve the following problems.

Problem 1. a) Let R be a UFD and let K be the field of fractions of R. Let $f(x) = f_0 + f_1x + ... + f_nx^n \in R[x]$. Suppose that $z \in K$ is a root of f. Write $z = a/b$ for some $a, b \in R$ such that $\gcd(a, b) = 1$. Prove that $a|f_0$ and $b|f_n$. Conclude that if f is monic then $z \in R$.

b) Prove that if $n \in \mathbb{Z}$ is not a $k-$th power of an integer then there are no rational numbers r such that $r^k = n$. (In other words, $\sqrt[k]{n}$ is irrational). Hint: Use a).

c) Which of the following polynomials have a root in $\mathbb{Q}$?

$$2x^5 + 7x^2 - 3,\ 3x^5 + 2x^4 + 6x^2 + x - 2,\ x^{2007} - 12x^{1974} - 2007x^{12} - 1.$$

Hint: Use a) to reduce to a finite number of cases and verify each case.

Problem 2. Prove that the following polynomials are irreducible:

a)

$$\frac{1}{5}x^6 + 6x^5 - 3x^3 + \frac{6}{5}x - 24 \text{ in } \mathbb{Q}[x].$$

b) $x^4 - 5$ in $\mathbb{Q}[i][x]$.

c) $f(x) = [(x + 2)^p - 2^p]/x$ in $\mathbb{Q}[x]$, where p is odd prime.

Problem 3. Let R be UFD with field of fractions K and let $f = f_0+f_1x+...+f_nx^n \in R[x]$. Suppose that there is a prime ideal P of R such that $f_n \notin P$, $f_0, ..., f_{n-1} \in P$ and $f_0 \notin P^2$. Prove that f is irreducible in the ring $K[x]$. Hint: Use Problem 2b) from homework 28.