Solution to Problem 6: Let $a = a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 + ... + a_n \cdot 10^n$.

Since $10^k \equiv 0 \pmod 2$ for $k > 0$ we see that

$$a \equiv a_0 \pmod 2 .$$

Thus $2|a$ iff $2|a_0$.

Similarly, $10^k \equiv 0 \pmod 5$ for $k > 0$ so

$$a \equiv a_0 \pmod 5 .$$

Thus $5|a$ iff $a_0 = 0$ or $a_0 = 5$ (recall that the numbers a_i are the digits of a, i.e. they all are in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$).

Now $10 \equiv 2 \pmod 4$ and $10^k \equiv 0 \pmod 4$ for $k \geq 2$ so

$$a \equiv a_0 + a_1 \cdot 10 \equiv a_0 + 2a_1 \pmod 4 .$$

It follows that $4|a$ iff $4|a_0 + 2a_1$.

Similarly, we have $10 \equiv 2 \pmod 8$, $10^2 \equiv 4 \pmod 8$ and $10^k \equiv 0 \pmod 8$ for $k \geq 3$. Thus

$$a \equiv a_0 + a_1 \cdot 10 + a_2 \cdot 10^2 \equiv a_0 + 2a_1 + 4a_2 \pmod 8 .$$

In particular, $8|a$ iff $8|a_0 + 2a_1 + 4a_2$.

Note now that $10 \equiv 1 \pmod 3$ and $10 \equiv 1 \pmod 9$. Thus $10^k \equiv 1 \pmod 3$ and $10^k \equiv 1 \pmod 9$ for every $k \geq 0$. It follows that

$$a \equiv a_1 + a_1 + ... + a_n \pmod 3$$

and

$$a \equiv a_1 + a_1 + ... + a_n \pmod 9 .$$

In particular, $3|a$ iff $3|a_1 + a_1 + ... + a_n$ and $9|a$ iff $9|a_1 + a_1 + ... + a_n$.

Since $10 \equiv -1 \pmod{11}$, we have $10^k \equiv 1 \pmod{11}$ for k even and $10^k \equiv -1 \pmod{11}$ for k odd. Consequently,

$$a \equiv a_0 - a_1 + a_2 - a_3 + ... \pmod{11} .$$

Thus $11|a$ iff $11|a_0 - a_1 + a_2 - a_3 + ...$

Solution to Problem 12: We perform Euclidean algorithm to 89 and 55:

$$89 = 55 + 34,$$

$$55 = 34 + 21,$$

$$34 = 21 + 13,$$

$$21 = 13 + 8,$$

$$13 = 8 + 5,$$

$$8 = 5 + 3,$$

$$5 = 3 + 2,$$

$$3 = 2 + 1,$$

$$2 = 2 \cdot 1 + 0.$$

Thus

$$1 = 3-2 = 3-(5-3) = 2\cdot3-5 = 2(8-5)-5 = 2\cdot8-3\cdot5 = 2\cdot8-3\cdot(13-8) = 5\cdot8-3\cdot13 =$$

$$= 5 \cdot (21 - 13) - 3 \cdot 13 = 5 \cdot 21 - 8 \cdot 13 = 5 \cdot 21 - 8 \cdot (34 - 21) = 13 \cdot 21 - 8 \cdot 34 =$$

$$13 \cdot (55 - 34) - 8 \cdot 34 = 13 \cdot 55 - 21 \cdot 34 = 13 \cdot 55 - 21 \cdot (89 - 55) = 34 \cdot 55 - 21 \cdot 89.$$

Thus $\lambda = -21$, $\mu = 34$ work. From $1 = 34 \cdot 55 - 21 \cdot 89$ we see that

$$89 \cdot (-21) \equiv 1 \pmod{55} .$$

Mulyplying this congruence by 7 we get

$$89 \cdot (-21) \cdot 7 \equiv 7 \pmod{55} .$$

To simplyfy, note that $(-21) \cdot 7 = -147 \equiv 18 \pmod{55}$, so

$$89 \cdot 18 \equiv 7 \pmod{55} .$$

All solutions are given by $x = 18 + k \cdot 55$, $k \in \mathbb{Z}$.