Problem 1. Let R be a UFD. For a non-zero polynomial $f \in R[x]$ define the content $\text{cont}(f)$ of f as a greatest common divisor of the coefficients of f (note: this is only defined up to invertible element of R; in other words, the content is not a unique element of R but a class of associated elements). Prove that $\text{cont}(fg) = \text{cont}(f)\text{cont}(g)$. Hint: Use Gauss Lemma.

Solution: Note that by the very definition of $\text{cont}(f)$, we have $f = \text{cont}(f)F$ for some primitive polynomial F. Similarly $g = \text{cont}(g)G$ with G primitive. It follows that $fg = \text{cont}(f)\text{cont}(g)FG$. By Gauss' Lemma, FG is primitive and therefore $\text{cont}(fg) = \text{cont}(f)\text{cont}(g)$.

Problem 2. Let R be a UFD with a field of fractions K. Suppose that $f \in R[x]$ is monic. Prove that if $g \in K[x]$ is monic and $g|f$ in $K[x]$ then $g \in R[x]$.

Solution: Suppose that $f = gh$ for some $h \in K[x]$. There is $k \in K$ such that $kg \in R[x]$ is primitive. Thus $f = (kg)(k^{-1}h)$ in $K[x]$. By one of our results (Theorem 2), we conclude that $k^{-1}h \in R[x]$. So $f = (kg)(k^{-1}h)$ is a factorization in $R[x]$. Since f is monic, the leading coefficients of kg and $k^{-1}h$ are invertible in R. But the leading coefficient of kg is k (since g is monic), so $k \in R$ is invertible in R. Since $kg \in R[x]$, also $k^{-1}(kg) = g \in R[x]$.

Problem 3. Let $K \subseteq L$ be fields. Suppose that $f, g \in K[x]$ and $f|g$ in the ring $L[x]$. Prove that $f|g$ in the ring $K[x]$.

Solution: The division algorithm in $K[x]$ allows us to write $g = hf + r$ for some $h, r \in K[x]$ with $\deg r < \deg f$. This remains a true equality in the larger ring $L[x]$. But since $f|g$ in $L[x]$, we conclude that $f|(g - hf) = r$ in $L[x]$. Since $\deg r < \deg f$, this is only possible if $r = 0$. Thus $f|g$ in $K[x]$.