Problem 1. Let G be a group such that $g^2 = e$ for every $g \in G$. Prove that G is abelian.

Solution: Let $g, h \in G$. We have

$$e = (g * h)^2 = (g * h) * (g * h) = g * (h * g) * h.$$

We also have

$$e = g^2 * h^2 = (g * g) * (h * h) = g * (g * h) * h.$$

Thus

$$g * (h * g) * h = g * (g * h) * h$$

and by cancellatuin we arrive at $h * g = g * h$. This shows that G is abelian.

Problem 2. Let a, b be real numbers, $a \neq 0$. Define the map $T_{a,b} : \mathbb{R} \longrightarrow \mathbb{R}$ by $T_{a,b}(x) = ax + b$. Let G be the set off all such maps, i.e.

$$G = \{T_{a,b} : a, b \in \mathbb{R},\ a \neq 0\}.$$

a) Prove that G with composition of functions as $*$ is a group.

b) Suppose that $z \neq 0$. Show that $T_{a,b}$ and $T_{1,z}$ commute iff $a = 1$. .

Solution: a) First we need to check that G is closed under composition. We have

$$(T_{a,b} * T_{c,d})(x) = T_{a,b}(cx + d) = a(cx + d) + b = acx + ad + b = T_{ac,ad+b}(x)$$

so

$$T_{a,b} * T_{c,d} = T_{ac,ad+b}. \quad (1)$$

Thus G is closed under composition. Composition of functions is an associaive operation so the associativity axiom is satisfied automatically. According to (1), we have $T_{a,b} * T_{1,0} = T_{a,b} = T_{1,0} * T_{a,b}$ so $e = T_{1,0}$. Finally, given $T_{a,b}$ we look for $T_{x,y}$ such that $T_{x,y} * T_{a,b} = e = T_{1,0}$. By (1), $T_{x,y} * T_{a,b} = T_{xa,ab+y}$ so we want $xa = 1$ and $xb + y = 0$, i.e. $x = a^{-1}$ and $y = -a^{-1}b$. Thus $T_{a^{-1},-a^{-1}b}$ is the inverse of $T_{a,b}$. This completes our proof that G is a group.

b) By (1), $T_{a,b} * T_{1,z} = T_{a,az+b}$ and $T_{1,z} * T_{a,b} = T_{a,b+z}$. Thus $T_{a,b}$ and $T_{1,z}$ commute iff $az + b = b + z$, i.e. $(a - 1)z = 0$. Since $z \neq 0$, the last condition is equivalent to $a = 1$.