Problem 1. Let G be a group and let $H = \{g \in G : g^2 = e\}$.

a) Prove that if G is abelian then H is a subgroup of G.

b) Is H a subgroup when $G = D_8$ is the dihedral group of order 8?

Solution: a) Clearly $e \in H$. Let $a, b \in H$. Then

$$(a * b)^2 = (a * b) * (a * b) = a * (b * a) * b = a * (a * b) * b = a^2 * b^2 = e * e = e$$

so $a * b \in H$. Also, $(a^{-1})^2 = (a^2)^{-1} = e^{-1} = e$, so $a^{-1} \in H$. This proves that H is a subgroup.

b) Recall that $D_8 = \{I, T, T^2, T^3, S, ST, ST^2, ST^3\}$. Now $H = \{I, T^2, S, ST, ST^2, ST^3\}$. In particular, S, ST are in H but $S(ST) = S^2T = T$ is not, so H is not a subgroup.

Problem 2. Let G be the set of all bijections $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ which preserve distance, i.e. such that $|f(i) - f(j)| = |i - j|$ for all integers i, j.

a) Show that G is a subgroup of $\mathrm{Sym}(\mathbb{Z})$. It is called the **infinite dihedral group** and it is often denoted by D_∞.

b) The group G contains elements T, S such that $T(a) = a + 1$ and $S(a) = -a$ for all integers a. Prove that $S * T = T^{-1} * S$. Show that the subgroup $< T >$ is infinite. What is $< S >$?

c) Show that if $F \in G$ and $F(0) = 0$ then either $F = 1$ (the identity) or $F = S$.

d) Show that every element of G is of the fo T^i or $S * T^i$ for some integer i (try to use similar argument to the one we used for dihedral group of order n).

e) Suppose that $T^5 * S^7 * T^3 = S^a * T^b$. Find a and b.

Solution: a) Let $f, g \in G$ and $i, j \in \mathbb{Z}$. Then

$$|(f * g)(i) - (f * g)(j)| = |f(g(i)) - f(g(j))| = |g(i) - g(j)| = |i - j|$$

so $f * g \in G$. Also,

$$|i - j| = |f(f^{-1}(i)) - f(f^{-1}(j))| = |f^{-1}(i) - f^{-1}(j)|$$

so $f^{-1} \in G$. Clearly the identity I is in G, so G is a subgroup of $\mathrm{Sym}(\mathbb{Z})$.

b) Note that $T^{-1}(a) = a - 1$ for all $a \in \mathbb{Z}$. Thus

$$(S * T)(a) = S(T(a)) = S(a+1) = -(a+1) = -a - 1$$

and

$$(T^{-1} * S)(a) = T^{-1}(S(a)) = T^{-1}(-a) = -a - 1$$

for all $a \in \mathbb{Z}$. It follows that $S * T = T^{-1} * S$.

Observe now that for any integer m we have $T^m(0) = m$. It follows that if $m \neq n$ then $T^m \neq T^n$ so $< T >$ is infinite. Since $S^2 = I$, we have $< S >= \{I, S\}$.

c) Since $|F(1)| = |F(0) - F(1)| = |0 - 1| = 1$, we kave $F(1) = 1$ or $F(1) = -1$.

Suppose first that $F(1) = 1$. Let $n > 0$. Note that n is the only integer whose distance from 0 is n and whose distance from 1 is $n - 1$. But $|F(n) - 0| = |F(n) - F(0)| = |n - 0| = n$ and $|F(n) - 1| = |F(n) - F(1)| = |n - 1| = n - 1$. It follws that $F(n) = n$ for all positive integers n. Similarly, if $n < 0$, then n is the only integer whose distance from 0 is $|n|$ and whose distance from 1 is $|n| + 1$. Since $|F(n) - 0| = |F(n) - F(0)| = |n - 0| = |n|$ and $|F(n) - 1| = |F(n) - F(1)| = |n - 1| = |n| + 1$, we see that $F(n) = n$. This proves that $F = I$.

Suppose now that $F(1) = -1$. Then $(S * F)(0) = S(F(0)) = S(0) = 0$ and $(S * F)(1) = S(F(1)) = S(-1) = 1$. We just showed that this forces the equality $SF = I$, i.e. $F = S^{-1} = S$.

d) Let $G(0) = i$. Note that $T^i(0) = i$. Thus $(T^{-i} * G)(0) = 0$. By part c), we have either $T^{-i} * F = I$ or $T^{-1} * F = S$. In the former case, $F = T^i$ and in the latter case $F = T^i * S = S * T^{-i}$.

e) Note that $S^7 = S$ and $T^m * S = S * T^{-m}$ for any integer m. Thus

$$T^5 * S^7 * T^3 = T^5 * S * T^3 = S * T^{-5} * T^3 = S * T^{-2}.$$

Problem 3. In the group $GL_2(\mathbb{C})$ of all invertible 2×2 matrices with entries in complex numbers consider the matrices $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $k = ij = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. Let Q_8 be the set $\{I, -I, i, -i, j, -j, k, -k\}$.

a) Show that Q_8 is a subgroup of $GL_2(\mathbb{C})$. Write the table of multiplication in Q_8. Q_8 is called the **quaternion group**.

b) List all subgroups of Q_8.

Solution: a) Note that:

- $I \cdot a = a \cdot I = a$ for all $a \in Q_8$ (i.e. I is the identity in Q_8);
- $(-I)a = a(-I) = -a$ for all $a \in Q_8$;
- $ii = (-i)(-i) = jj = (-j)(-j) = kk = (-k)(-k) = -I$;
- $ij = (-i)(-j) = (-j)i = j(-i) = k$; $ji = (-j)(-i) = (-i)j = i(-j) = -k$;
- $jk = (-j)(-k) = (-k)j = k(-j) = i$; $kj = (-k)(-j) = (-j)k = j(-k) = -i$;
- $ki = (-k)(-i) = (-i)k = i(-k) = j$; $ik = (-i)(-k) = (-k)i = k(-i) = -j$;
- $i^{-1} = -i$, $j^{-1} = -j$, $k^{-1} = -k$.

This shows that Q_8 is closed under multiplication and inverses, so it is a subgroup of $GL_2(\mathbb{C})$.

b) The subgroups of Q_8 are: $< I >= \{I\}$, $< -I >= \{-I, I\}$, $< i >= \{I, i, -I, -i\}$, $< j >= \{I, j, -I, -j\}$, $< k >= \{I, k, -I, -k\}$ and Q_8. In particular, every proper subgroup of Q_8 is cyclic.