Problem 1. Let G be a group and let H, K be two subgroups of G such that $H \cup K$ is also a subgroup. Prove that either $H \subseteq K$ or $K \subseteq H$.

Solution: Suppose that neither $H \subseteq K$ nor $K \subseteq H$. Then there is $h \in H$, $h \notin K$ and there is $k \in K$, $k \notin H$. Since $H \cup K$ is a subgroup, we have $hk \in H \cup K$. It follows that $hk \in H$ or $hk \in K$. In the former case, $k = h^{-1}(hk) \in H$ and the latter case $h = (hk)k^{-1} \in K$. In both cases we get a contradiction.

Problem 2. Let G be a group. Define the **center** of G as the subset $Z(G)$ of all elements which commute with every element of G, i.e.

$$Z(G) = \{g \in G : ag = ga \text{ for every } a \in G\}.$$

a) Prove that $Z(G)$ is a subgroup of G.

b) Find $Z(D_6)$, $Z(D_8)$, $Z(Q_8)$ and $Z(D_\infty)$.

c) What is $Z(D_{2n})$?

Solution: a) Clearly $e \in G$. Let $a, b \in Z(G)$. For any $g \in G$ we have

$$(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab)$$

so $ab \in Z(G)$. Also

$$a^{-1}g = (a^{-1}g)(aa^{-1}) = a^{-1}(ga)a^{-1} = (a^{-1}a)ga^{-1} = ga^{-1}$$

so $a^{-1} \in G$. Thus G is a subgroup of G.

b) Recall that $D_6 = \{I, T, T^2, S, ST, ST^2\}$. Note that $S(S^aT^b) = S^{a+1}T^b$ and $(S^aT^b)S = S^a(T^bS) = S^aST^{-b} = S^{a+1}T^{-b}$. Thus $S(S^aT^b) = (S^aT^b)S$ iff $T^b = T^{-b}$, i.e. $T^{2b} = I$. This implies that $b = 0$. Thus the only elements which commute with S are I and S (i.e the centralizer $C(S) = \{I, S\}$). Note that S does not commute with T. Since elements of $Z(D_6)$ commute with all elements of D_6, we have $Z(D_6) = \{I\}$.

Similarly, $D_8 = \{I, T, T^2, T^3, S, ST, ST^2, ST^3\}$. As before, $S(S^aT^b) = S^{a+1}T^b$ and $(S^aT^b)S = S^a(T^bS) = S^aST^{-b} = S^{a+1}T^{-b}$, so $S(S^aT^b) = (S^aT^b)S$ iff $T^b = T^{-b}$, i.e. $T^{2b} = I$. This holds iff $b = 0$ or $b = 2$ (we only consider $b \in \{0, 1, 2, 3\}$). Thus the only elements which commute with S are I, S, T^2, ST^2. Note that neither S nor ST^2 commute with T. This implies that $Z(D_8) \subseteq \{I, T^2\}$. On the other hand, T^2 commutes with S and T so it commutes with all elements in D_8. Thus $Z(D_8) = \{I, T^2\}$.

Recall that $Q_8 = \{I, -I, i, -i, j, -j, k, -k\}$ (see Problem 3 of homework 32). Note that $\pm i$ does not comute with $\pm j$ and it does not commute with $\pm k$. Thus none of the elements $i, -i, j, -j, k, -k$ is in the center of Q_8. On the other hand, both I and $-I$ commute with all elements in Q_8. Thus $Z(Q_8) = \{I, -I\}$.

In D_∞ we have $T^m S = ST^{-m} \neq ST^m$ for $m \neq 0$, so neither S nor any non-trivial power of T belong to $Z(D_\infty)$. Furthermore, $(ST^m)T = ST^{m+1}$ and $T(ST^m) = (TS)T^m = (ST^{-1})T^m = ST^{m-1} \neq (ST^m)T$. Thus no element of the form ST^m belongs to $Z(D_\infty)$. It follows that the only element in $Z(D_\infty)$ is I, i.e. $Z(D_\infty) = I$.

c) As in b) we first find the centralizer of S. Since $S(S^aT^b) = S^{a+1}T^b$ and $(S^aT^b)S = S^a(T^bS) = S^aST^{-b} = S^{a+1}T^{-b}$, we have $S(S^aT^b) = (S^aT^b)S =$ iff $T^b = T^{-b}$ iff $T^{2b} = I$. In n is odd this holds only when $b = 0$ and when n is even then $b = 0$ or $b = n/2$. We see that

$$C(S) = \begin{cases} \{I, S\} & \text{if } n \text{ is odd;} \\ \{I, S, T^{n/2}, ST^{n/2}\} & \text{if } n \text{ is even.} \end{cases}$$

Note also that neither S nor $ST^{n/2}$ commute with T. Thus if n is odd then $Z(D_{2n}) = \{I\}$. For n even, $T^{n/2}$ commutes with all elements if D_{2n} (since it commutes with S and T) so $Z(D_{2n}) = \{I, T^{n/2}\}$ for n even.

Problem 3. Let G be a group and let H, K be subgroups of G.

a) Show that $H \cap K$ is a subgroup of G.

b) Suppose that $h_1, h_2 \in H$. Prove that $h_1(H \cap K) = h_2(H \cap K)$ iff $h_1K = h_2K$. Conclude that $[H : H \cap K] \leq [G : K]$.

c) Let L be a subgroup of H. Suppose that $[G : L]$ is finite. Prove that $[G : H]$ and $[H : L]$ are finite and $[G : L] = [G : H][H : L]$. Hint: Show that if $aH \neq bH$ then $aL \neq bL$ and that H/L is a subset of G/L.

d) Suppose that $[G : H]$ and $[G : K]$ are finite. Prove that $[G : H \cap K] \leq [G : H][G : K]$ (so, in particular, $[G : H \cap K]$ is finite).

Solution: a) Since $e \in K$ and $e \in H$, we have $e \in H \cap K$. If $a, b \in H \cap K$ then $a, b \in H$ and $a, b \in K$. Thus $ab \in H$ and $ab \in K$and $a^{-1} \in H$ and $a^{-1} \in K$. It follows that $ab \in H \cap K$ and $a^{-1} \in H \cap K$. This proves thta $H \cap K$ is a subgroup.

b) Note that $h_1(H \cap K) = h_2(H \cap K)$ iff $h_2^{-1}h_1 \in H \cap K$ iff $h_2^{-1}h_1 \in K$ (since we know that $h_2^{-1}h_1 \in H$), iff $h_1K = h_2K$. It follows that the map $\Phi : H/(H \cap K) \longrightarrow G/K$ given by $\Phi(h(H \cap K)) = hK$ is well defined and injective. Thus $[H : H \cap K] \leq [G : K]$.

c) Note that if $aL = bL$ then $b^{-1}a \in L \subseteq H$ so $aH = bH$. It follows that the map $G/L \longrightarrow G/H$ given by $aL \mapsto aH$ is well defined and clearly surjective. Thus G/H is finite. Clearly H/L is a subset of G/L consisting of those cosets of L which are of the form hL for some $h \in H$. Thus H/L is finite.

Suppose now that $g_1H, ..., g_sH$ are different cosets of H in G, $s = [G : H]$. Similarly, let $h_1L, ..., h_tL$ be different cosets of L in H, $t = [H : L]$. We claim that the cosets g_it_jL are pairwise distinct and give all cosests of L in G. In fact, if $g_it_jL = g_ut_vL$ then $t_v^{-1}g_u^{-1}g_it_j \in L$. Since $L \subseteq H$ and $t_j, t_v \in H$, we get $g_u^{-1}g_i \in H$, so $g_iH = g_uH$ and therefore $i = u$. Thus $t_v^{-1}g_u^{-1}g_it_j = t_v^{-1}t_j \in L$, so $t_jL = t_vL$ and therefore $j = v$. This proves that the cosets g_it_jL, $i = 1, 2, ..., s; j = 1, 2, ..., t$ are pairwise distinct. If $g \in G$ then $g \in g_iH$ for sime i, so $g = g_ih$ for some $h \in H$. Furthermore, $h \in h_jL$ for some j, so $h^{-1}h_j \in L$. We see that $g^{-1}(g_ih_j) = h^{-1}g_i^{-1}g_ih_j = h^{-1}h_j \in L$, i.e. $gL = (g_ih_j)L$. This proves that each coset of L in G is equal to one of $(g_ih_j)L$. It follows that G/L has st elements, i.e. $[G : L] = [G : H][H : L]$.

d) By b) we have $[H : H \cap K]$ is finite and $[H : H \cap K] \leq [G : K]$. Taking $L = H \cap K$ in c) we see that $[G : H \cap K] = [G : H][H : H \cap K]$ is finite and $[G : H \cap K] \leq [G : H][G : K]$.