Homework
due on Monday, November 19

Read sections 2.4, 2.5, 2.6, 2.7, 2.8 in Lauritzen's book and sections 3.2.1, 3.2.2, 3.2.3, 3.3.1, 3.3.2, 3.3.3 in Cameron's book.
Solve the following problems.

Problem 1. Let G be a finite group of order mn, where $\gcd(m,n) = 1$. Suppose that Q is a subgroup of G of order m. Let N be a normal subgroup of G.

a) Show that if H is a subgroup of G then $|H|$ can be written uniquely as $\alpha\beta$, where $\alpha|m$ and $\beta|n$.

In particular, $|N| = st$ with $s|m$ and $t|n$ and $|QN| = uv$ with $u|m$ and $v|n$.

b) Prove that $mt||QN|$. Conclude that $m = u$ and $t|v$.

c) Prove that $|QN/N|$ divides m. Conclude that $t = v$. Hint: Use the third isomorphism theorem.

d) Prove that $|QN/N| = m/s$. Conclude that $|N \cap Q| = s$.

Remark. Note that in the notation above we have $n = [G : Q]$ (Lagrange's Theorem). Thus our assumption is that $\gcd(|Q|, [G : Q]) = 1$. Any subgroup Q which satisfies this condition is called a **Hall subgroup** of G. So the can summarise the result of this problem as follows: if Q is a hall subgroup of G and N is a normal subgroup then $Q \cap N$ is a Hall subgroup of N.

Problem 2. Let G and H be groups. On the set $G \times H$ define a multiplication by $(g,h)(a,b) = (ga, hb)$.

a) Prove that $G \times H$ with above defined multiplication is a group. It is called the **product** of G and H.

b) Prove that the maps $\pi_G : G \times H \longrightarrow G$ and $\pi_H : G \times H \longrightarrow H$ defined by $\pi_G(g,h) = g$ and $\pi_H(g,h) = h$ are homomorphisms. What are the kernels of these maps?

c) Suppose that $G = H$ and let $\Delta = \{(a,b) \in G \times G : a = b\}$. Prove that Δ is a subgroup of $G \times G$ which is isomorphic to G. Prove that Δ is normal iff G is abelian. If G is abelian, show that $G \times G/\Delta$ is isomorphic to G.

d) Let $f_1 : Q \longrightarrow G$ and $f_2 : Q \longrightarrow H$ be homomorphisms. Define $f : Q \longrightarrow G \times H$ by $f(a) = (f_1(a), f_2(a))$. Prove that f is a homomorphism and that $\ker f = \ker f_1 \cap \ker f_2$. Conclude that if A, B are normal subgroups of a group G then $G/(A \cap B)$ is isomorphic to a subgroup of $(G/A) \times (G/B)$.

Problem 3. Consider the dihedral group $D_{2n} = \{I, T, ..., T^{n-1}, S, ST, ..., ST^{n-1}\}$ of order $2n$. Let $f : D_{2n} \longrightarrow A$ be a homomorphism, where A is an abelian group.

a) Prove that $T^2 \in \ker f$.

b) Prove that if n is odd then the image of f has either one or two elements. Give example of a homomorphism f whith image of order 2.

c) Let n be even. Define a map $g : D_{2n} \longrightarrow \mathbb{Z}/2\mathbb{Z} \times Z/2\mathbb{Z}$ by

$$g(S^a T^b) = (a + 2\mathbb{Z}, b + 2\mathbb{Z}).$$

Prove that g is a surjective homomorphism and find its kernel.

d) Prove that if n is even then the image of f is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times Z/2\mathbb{Z}$ or $\mathbb{Z}/2$ or the trivial group.