Problem 1. a) Find the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 2 & 5 \end{pmatrix}$$

b) Find the cycle decomposition and order of

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix}$;
2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$;
3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix}$.

c) Express the following permutations as products of disjoint cycles and find their orders

1. $(1,2,3,4,5)(1,2,3,4,6)(1,2,3,4,7)$;
2. $(1,2,3)(3,4,5,6)(1,2,3)^{-1}$;
3. $(1,2,3,4,5,6)^3$.

Express each of these permutations as a product of transpositions.

d) Find a permutation $\rho \in S_n$ such that $\rho\tau\rho^{-1} = \sigma$ or prove that no such ρ exists, where

1. $\tau = (1,3,4)$, $\sigma = (2,3,5)$, $n = 5$;
2. $\tau = (1,4)(2,3,5)$, $\sigma = (2,4,1)(5,3)$, $n = 5$;
3. $\tau = (1,3,5,7)(2,4,6)$, $\sigma = (1,6)(4,5,7,3,2)$.

Solution: a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}.$$

b)

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 7 & 6 & 9 & 8 & 5 \end{pmatrix} = (1,3,4,2)(5,7,9)$. It has order 12;

2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} = (1,7)(2,6)(3,5)$. It has order 2;

3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 5 & 6 & 7 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 5 & 7 & 3 & 2 & 1 & 4 \end{pmatrix} =$ $(1,6)(2,5)(3,7,4)$. It has order 6.

c)

1. $(1,2,3,4,5)(1,2,3,4,6)(1,2,3,4,7) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{pmatrix} = (1,4,7,3,6,2,5) =$ $(1,5)(1,2)(1,6)(1,3)(1,7)(1,4)$; It has order 7.

2. $(1,2,3)(3,4,5,6)(1,2,3)^{-1} = (1,4,5,6) = (1,6)(1,5)(1,4)$. It has order 4.;

3. $(1,2,3,4,5,6)^3 = (1,4)(2,5)(3,5)$. It has order 2.

d)

1. Since $\tau = (1,3,4)$ and $\sigma = (2,3,5)$ have the same type of cycle decomposition, they are conjugete in S_n. Any ρ such that $\rho(1) = 2$, $\rho(3) = 3$ and $\rho(4) = 5$ will work. For example, $\rho = (1,2)(4,5)$.

2. Since $\tau = (1,4)(2,3,5)$ and $\sigma = (2,4,1)(5,3)$ have the same type of cycle decomposition, they are conjugete in S_n. Any ρ such that $\rho(1) = 5$, $\rho(4) = 3$, $\rho(2) = 2$, $\rho(3) = 4$, $\rho(5) = 1$ will work. Thus $\rho = (1,5)(4,3)$.

3. The permutatuins $\tau = (1,3,5,7)(2,4,6)$ and $\sigma = (1,6)(4,5,7,3,2)$ have different structure of cycle decomposition so they are not conjugate.

Problem 2. Let σ, τ be two transpositions in S_n, $n \geq 3$. Show that $\sigma\tau$ can be expressed as a product of $3-$cycles (not necessarily disjoint).

Solution: If σ, τ are disjoint then $\sigma = (a,b)$, $\tau = (c,d)$ and a,b,c,d are pairwise different. Then

$$(a,b)(c,d) = (a,b,c)(b,c,d).$$

If M_σ and M_τ have one element in common that we can write $\sigma = (a, b)$, $\tau = (a, c)$, where a, b, c are pairwise different. Thus

$$(a, b)(a, c) = (a, c, b).$$

Finally, if M_σ and M_τ have more than one element in common then $\sigma = \tau$ (note that both M_σ and M_τ have 2 elements) so $\sigma\tau = id = (1, 2, 3)(1, 2, 3)(1, 2, 3)$.