Problem 1. a) Which of the following permutations are even?

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix}$;

2. $(1,2,3,4,5,6)(7,8,9)$;

3. $(1,2)(1,2,3)(4,5)(5,6,8)(1,7,9)$.

b) Prove that a $k-$cycle is even iff k is odd.

c) In the even permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & & & 7 & 8 & 9 & 6 \end{pmatrix}$$

two entries are missing. Find the missing entries.

Solution: a) Checking whether a given permutation is odd or even directly from the definition is not the most efficient way. Instead we will use part b) and the fact that the sign of a permutation is a homomorphism.

1. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix} = (1,2,4)(3,5)(6,7,8,9)$. The sign of $(1,2,4)$ is 1 and the sign of both $(3,5)$ and $(6,7,8,9)$ is -1. Thus the sign of our permutation is $1(-1)(-1) = 1$, i.e. it is even.

2. The sign of the first factor of $(1,2,3,4,5,6)(7,8,9)$ is -1 and the second factor has sign 1, so the product has sign -1, i.e. it is odd;

3. The product $(1,2)(1,2,3)(4,5)(5,6,8)(1,7,9)$ has two odd permutations and three even permutations so it is an even permutation. .

b) Let $(a_1, a_2, ..., a_k)$ be a $k-$cycle. We have seen that

$$(a_1, a_2, ..., a_k) = (a_1, a_k)(a_1, a_{k-1})...(a_1, a_2)$$

is a product of $k-1$ transpositions. Thus $(a_1, a_2, ..., a_k)$ is odd if $k-1$ is odd and it is even if $k-1$ is even. This proves our claim.

c) One of the missing entries is 4 and the other is 5. If the first missing entry is 4 then our permutation is $(1,3,2)(6,7,8,9)$, which is an odd permutation. If the first missing entry is 5 then our permutation is $(1,3,2)(4,5)(6,7,8,9)$, which is even. Thus our permuation is equal to

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 5 & 4 & 7 & 8 & 9 & 6 \end{pmatrix}$$

Problem 2. Prove that every element of A_n, $n \geq 3$ is a product of $3-$cycles.

Solution: In the solution to Problem 2 of homework 38 we showed that a product of any two transpositions is a product of $3-$cycles. Since any even permutation is a product of an even number of transpositions, we see that an even permutation is a product of $3-$cycles (just pair the transpoitions).

Problem 3. a) Prove that the center of S_n is trivial for $n \geq 3$.

b) Find the number of conjugacy classes in S_6.

c) Prove that the centralizer of $(1, 2, ..., k)$ in S_n has $k(n-k)!$ elements.

Solution: a) Let τ be a non-trivial permutation. Thus $\tau(a) = b \neq a$ for some a. Since $n \geq 3$, there is c different form a and b. Note that $\tau(a,c)\tau^{-1} = (\tau(a), \tau(c)) = (b, \tau(c)) \neq (a,c)$. Thus τ and (a,c) do not commute, so τ is not in the center of S_n. This proves that the center of S_n is trivial.

b) Recall that two elements of S_n are conjugate iff they have the same type of cycle decomposition, i.e. for each k they have the same number of $k-$cycyles in their cycle decomposition. Now for $n = 6$, we have the following possible cycle decompositions:

1. one cycle of length 6;
2. one cycle of length 5
3. one cycle of length 4
4. one cycle of length 3
5. one cycle of length 2
6. two cycles, one of length 4 and one of length 2;

7. two cycles, one of length 3 and one of length 2;

8. two cycles, each of length 3;

9. two cycles, each of length 2;

10. three cycles, each of length 2.

Thus S_6 has 10 conjugacy classes.

c) Recall that for a permutation $\tau \in S_n$ we have $\tau(1,2,...,k)\tau^{-1} = (\tau(1), \tau(2), ..., \tau(k))$. Thus τ centralizes $(1,2,...,k)$ iff $(\tau(1), \tau(2), ..., \tau(k)) = (1,2,...,k)$. If $\tau(1) = i$ for $i \in \{1,2,...,k\}$ then the values of τ on $\{1,2,...,k\}$ are uniqually determined and its values on $\{k+1,...,n\}$ can be any permutation of $\{k+1,...,n\}$. Thus for each $i \in \{1,2,...,k\}$ we have $(n-k)!$ elements centralizing $(1,2,...,k)$ and taking 1 to i. Since i can assume k values, the centralizer of $(1,2,...,k)$ has $k(n-k)!$ elements.

Problem 4. Let H be a subgroup of S_n which contains $(1,2)$ and $(1,2,3,...,n)$. Prove that $H = S_n$. Hint: Show that H contains all transpositions.

Solution: Let $\tau = (1,2,3,...,n)$. Since H is a subgroup and both $\tau, (1,2)$ are in H, also $\tau^k(1,2)\tau^{-k} = (\tau^k(1), \tau^k(2) = (k+1, k+2) \in H$ for $k = 1,2,...,n-2$ and $\tau^{n-1}(1,2)\tau^{1-n} = (n,1) \in H$. Note now that for $s < k < n$ we have $(k,k+1)(s,k)(k,k+1) = (s,k+1)$. We claim that this allows us to show that $(i,j) \in H$ for any $i < j$. In fact, we have seen that $(i,i+1) \in H$ and if $(i,j) \in H$ with $i < j < n$ then also $(i,j+1) \in H$. Thus H contains all transpositions. Since every permutation is a product of transpositions, we see that H contains all permutations, i.e. $H = S_n$.