Problem 1. Suppose that a group G acts on a set X. For $g \in G$ define a function $l_g : X \longrightarrow X$ by $l_g(x) = g * x$.

a) Prove that l_g is a bijection (so it belongs to $\mathrm{Sym}(X)$).

b) Prove that the map $G \longrightarrow \mathrm{Sym}(X)$, $g \mapsto l_g$ is a homomorphism.

c) Suppose that $\psi : G \longrightarrow \mathrm{Sym}(X)$ is a homomorphism. Define a function $* : G \times X \longrightarrow X$ by $g * x = \psi(g)(x)$. Prove that $*$ is a group action of G on X.

d) Show that b) and c) define inverse of each other bijections between actions of G on X and homomorphisms $G \longrightarrow \mathrm{Sym}(X)$.

Solutuion: a) Note that

$$(l_{g^{-1}}l_g)(x) = l_{g^{-1}}(l_g(x)) = g^{-1} * (g * x) = (g^{-1}g) * x = e * x = x$$

and similarly, $(l_g l_{g^{-1}})(x) = x$ for all $x \in X$. Thus l_g and l_{g-1} are inverse of each other bijections.

b) Note that $l_{gh}(x) = (gh) * x = g * (h * x) = l_g(l_h(x)) = (l_g l_h)(x)$. This proves that $g \mapsto l_g$ is a homomorphism.

c) We have

$$g * (h * x) = \psi(g)(\psi(h)(x)) = (\psi(g)\psi(h))(x) = \psi(gh)(x) = (gh) * x$$

and

$$e * x = \psi(e)(x) = id(x) = x.$$

(here id is the identity bijection of X). This means that $*$ is an action of G on X.

d) Starting with an action $*$ we get a homomorphism $g \mapsto l_g$ which then defines an action $\bullet$ by $g \bullet x = l_g(x) = g * x$ so we get back the original action $*$. Conversely, starting with a homomorphism ψ we get an action $*$ which then defines a homomorpgism $g \mapsto l_g$, where $l_g(x) = g * x = \psi(g)(x)$, so $l_g = \psi(g)$. Thus b) and c) define inverse of each other bijections between actions of G on X and homomorphisms $G \longrightarrow \mathrm{Sym}(X)$.

Problem 2. Let G be a group and H a subgroup of G. Let $X = G/H$ be the set of all left cosets of H in G. Define $* : G \times X \longrightarrow X$ by $g * (aH) = (ga)H$.

Prove that $*$ is an action of G on X. Show that this action has only one orbit (such action is called **transitive**). According to Problem 1 this action corresponds to a homomorphism $G \longrightarrow \mathrm{Sym}(X)$. Prove that kernel of this homomorphism is the largest normal subgroup of G which is contained in H (i.e. that any normal subgroup of G which is contained in H is also conatined in this kernel).

Solution: First we check that $*$ is well defined: if $aH = bH$ then $b^{-1}a \in H$. But $b^{-1}a = (gb)^{-1}(ga)$, so $gaH = gbH$. This means that $*$ is well defined.

Clearly, $g * (h * aH) = g * haH = ghaH = gh * (aH)$ and $e * aH = eaH = aH$, so $*$ is an action. For any $a, b \in G$ we have $bH = ba^{-1}aH = (ba^{-1}) * aH$, so any two elements of X are in the same orbit. This means that there is only one orbit, i.e. the action is transitive.

Conisder now the homomorphism $G \longrightarrow \mathrm{Sym}(X)$ defined by this action. Let K be the kernel of this homomorphism. This is a normal subgroup of G which consists of all elements $g \in G$ such that $gaH = aH$ for every $a \in G$. In other words, $g \in K$ iff $a^{-1}ga \in H$ for all $a \in G$. In particular, for $a = e$ we get $g \in H$, so $K \subseteq H$. Let L be a normal subgroup of G contained in H. If $a \in G$ and $g \in L$ then $a^{-1}ga \in L \subseteq H$. Thus $g \in K$. This proves that $L \subseteq K$. In other words, K is the largest normal subgroup of G which is contained in H.

Problem 3. Suppose that a group G acts on a set X. Let k be a positive integer, $k \leq |X|$. Let $P_k(X)$ be the set of all subsets of X which have k elements. For $g \in G$ and $A = \{x_1, x_2, ..., x_k\} \in P_k(X)$ define $g * A = \{g * x_1, g * x_2, ..., g * x_k\}$. Prove that this defines an action of G of $P_k(X)$.

Solution: Note that $g * A = l_g(A)$, where l_g is defined in Problem 1. Since l_g is a bijection, $l_g(A)$ and A have the same number of elements, so $*$ is well defined. Clearly $e * A = l_e(A) = A$ and

$$g * (h * A) = l_g(l_h(A)) = (l_g l_h)(A) = l_{gh}(A) = (gh) * A.$$

Thus $*$ is an action of G on X.