Problem 1. a) Find all Sylow 2−subgroups of S_4. Hint. You may consult Problem 4 b) from Test III.

b) What is the number of Sylow 3−subgroups of S_4?

Solution: a) In the solution to Problem 4 b) from Test III (see also Problem 7 of Test III) we have seen that S_4 has a subgroup of order 8 isomorphic to D_8, namely

$$P = \{e, (1,2,3,4) = T, (1,3)(2,4) = T^2, (1,4,3,2) = T^3,$$

$$(2,4) = S, (1,4)(2,3) = ST, (1,3) = ST^2, (1,2)(3,4) = ST^3\}.$$

We have also seen that is is not normal. It follows that $t_2 > 1$. Since $t_2|3$, we have $t_2 = 3$, i.e. S_4 has 3 Sylow 2−subgroups. By Sylow Theorem we know that all of them are conjugate. Note that

$$Q = (1,4)P(1,4)^{-1} = \{e, (4,2,3,1), (4,3)(2,1), (4,1,3,2),$$

$$(2,1), (1,4)(2,3), (4,3), (4,2)(3,1)\}$$

is a second Sylow 2−subgroup of S_4 and

$$R = (1,2)P(1,2)^{-1} = \{e, (2,1,3,4), (2,3)(1,4), (2,4,3,1),$$

$$(1,4), (2,4)(1,3), (2,3), (1,2)(3,4)\}$$

is third. Thus P, Q, R are the Sylow 2−subgroups of S_4.

b) Note that a Sylow 3− subgroup of S_4 has order 3. Thus $P = \{e, (1,2,3), (1,3,2)\}$ $=< (1,2,3) >$ is a Sylow 3−subgroup of S_4 and $Q = \{e, (1,2,4), (1,4,2)\} =< (1,2,4) >$ is another one. Thus $t_3 > 1$. Now $t_3 \equiv 1 \pmod 3$ and $t_3|8$ by Solow Theorem. The only possibility for t_3 is then $t_3 = 4$. It is easy to list the remaining two Sylow 3−subgroups, namely $R = \{e, (2,3,4), (2,4,3)\} =< (2,3,4) >$ and $S = \{e, (1,3,4), (1,4,3)\} =< (1,3,4) >$.

Problem 2. Let G be a finite group and let p be a prime divisor of $|G|$.

a) Suppose that G has a subgroup H such that $[G : H] < p$. Prove that G is not simple. Hint: Consider the action of G on the set X of left cosets of H described

in Problem 2 of homework 40. Prove that the corresponding homomorphism $G \longrightarrow \mathrm{Sym}(X)$ is not injective.

b) Suppose that p is the smallest prime divisor of $|G|$ and that $[G : H] = p$. Prove that H is normal in G. Hint: Consider the homomorphism suggested in a) and prove that H is its kernel.

Solution: a) As discussed in problem 2 of homework 40, the action of G on the set X of left cosets of H in G corresponds to a homomorphism $\psi : G \longrightarrow \mathrm{Sym}(X)$. Let $m = |X| = [G : H]$, so $m < p$. Thus $\mathrm{Sym}(X)$ is isomorphic to the symmetric group S_m. Note that $p||G|$ and $p \nmid |\mathrm{Sym}(X)| = m!$. Thus ψ can not be injective (otherwise, the image of ψ would be a subgroup of $\mathrm{Sym}(X)$ of order divisible by p, which is not possible). Thus $\ker\psi$ is a non-trivial normal subgroup of G. Since we have seen in Problem 2 of homework 40 that $\ker\psi \subseteq H$, $\ker\psi$ is also a proper subgroup. Thus G is not simple.

b) We keep the notation introduced in a), so now $m = p$. Let $K = \ker\psi$. By the First Isomorphism Theorem, the image of ψ is isomorphic to G/K and it is a subgroup of $\mathrm{Sym}(X)$, so its order divides $p!$. It also divides $|G|$, so $|G/K|$ divides $\gcd(|G|, p!)$. Since all prime divisors of $|G|$ are larger or equal to p, we have $\gcd(|G|/p, (p-1)!) = 1$ so $\gcd(|G|, p!) = p$. Thus $|G/K|$ divides p, so it is equal to p. Thus $[G : K] = p = [G : H]$. Since $K \subseteq H$, we must have $H = K$. Thus H is normal in G.

Problem 3. Let G be a finite group and let P be a Sylow $p-$subgroup of G (for some prime p).

a) Suppose that $f : G \longrightarrow H$ is a surjective homomorphism. Prove that $f(P)$ is a Sylow $p-$subgroup of H.

b) Let N be a normal subgroup of G. Prove that $P \cap N$ is a Sylow $p-$subgroup of N.

Solution: a) Let $|G| = p^a m$, $p \nmid m$, $K = \ker f$ and $|K| = p^b n$, $p \nmid n$. Since $|K|||G|$, we have $b \leq a$ and $n|m$. By the First Isomorphism Theorem, H is isomorphic to G/K, so $|H| = p^{a-b}m/n$. Again by the First Isomorphism Theorem, $f(P)$ is isomorphic to $P/(P \cap K)$. Since $P \cap K$ is a $p-$subgroup of K, $|P \cap K| = p^c$ for some $c \leq b$. Thus $|f(P)| = p^{a-c}$. On the other hand, $p^{a-c} = |f(P)|||H|$, so $a - c \leq a - b$,

i.e. $c \geq b$. Thus $c = b$ and $|f(P)| = p^{a-b}$, i.e. $f(P)$ is a Sylow $p-$subgroup of H.

b) Apply the solution to a) to the canonical homomorphism $f : G \longrightarrow G/N$, so $N = \ker f = K$. We have seen that $|P \cap K| = p^b$, where $|K| = p^b n$, $p \nmid n$. Thus $P \cap K = P \cap N$ is a Sylow $p-$subgroup of N.

Remark. Look also at solution to Problem 1 of Homework 37.