Problem 1. a) Find all Sylow 2−subgroups of $S_4$. Hint. You may consult Problem 4 b) from Test III.

b) What is the number of Sylow 3−subgroups of $S_4$?

Solution: a) In the solution to Problem 4 b) from Test III (see also Problem 7 of Test III) we have seen that $S_4$ has a subgroup of order 8 isomorphic to $D_8$, namely

$$P = \{ e, (1, 2, 3, 4) = T, (1, 3)(2, 4) = T^2, (1, 4, 3, 2) = T^3, (2, 4) = S, (1, 4)(2, 3) = ST, (1, 3) = ST^2, (1, 2)(3, 4) = ST^3 \}.$$  

We have also seen that is is not normal. It follows that $t_2 > 1$. Since $t_2 | 3$, we have $t_2 = 3$, i.e. $S_4$ has 3 Sylow 2−subgroups. By Sylow Theorem we know that all of them are conjugate. Note that

$$Q = (1, 4)P(1, 4)^{-1} = \{ e, (4, 2, 3, 1), (4, 3)(2, 1), (4, 1, 3, 2), (2, 1), (1, 4)(2, 3), (4, 3), (4, 2)(3, 1) \}$$

is a second Sylow 2−subgroup of $S_4$ and

$$R = (1, 2)P(1, 2)^{-1} = \{ e, (2, 1, 3, 4), (2, 3)(1, 4), (2, 4, 3, 1), (1, 4), (2, 4)(1, 3), (2, 3), (1, 2)(3, 4) \}$$

is third. Thus $P, Q, R$ are the Sylow 2−subgroups of $S_4$.

b) Note that a Sylow 3−subgroup of $S_4$ has order 3. Thus $P = \{ e, (1, 2, 3), (1, 3, 2) \} = < (1, 2, 3) >$ is a Sylow 3−subgroup of $S_4$ and $Q = \{ e, (1, 2, 4), (1, 4, 2) \} = < (1, 2, 4) >$ is another one. Thus $t_3 > 1$. Now $t_3 \equiv 1 \pmod{3}$ and $t_3 | 8$ by Sylow Theorem. The only possibility for $t_3$ is then $t_3 = 4$. It is easy to list the remaining two Sylow 3−subgroups, namely $R = \{ e, (2, 3, 4), (2, 4, 3) \} = < (2, 3, 4) >$ and $S = \{ e, (1, 3, 4), (1, 4, 3) \} = < (1, 3, 4) >$.

Problem 2. Let $G$ be a finite group and let $p$ be a prime divisor of $|G|$.

a) Suppose that $G$ has a subgroup $H$ such that $[G : H] < p$. Prove that $G$ is not simple. Hint: Consider the action of $G$ on the set $X$ of left cosets of $H$ described
in Problem 2 of homework 40. Prove that the corresponding homomorphism $G \longrightarrow \text{Sym}(X)$ is not injective.

b) Suppose that $p$ is the smallest prime divisor of $|G|$ and that $[G : H] = p$. Prove that $H$ is normal in $G$. Hint: Consider the homomorphism suggested in a) and prove that $H$ is its kernel.

**Solution:** a) As discussed in problem 2 of homework 40, the action of $G$ on the set $X$ of left cosets of $H$ in $G$ corresponds to a homomorphism $\psi : G \longrightarrow \text{Sym}(X)$. Let $m = |X| = [G : H]$, so $m < p$. Thus $\text{Sym}(X)$ is isomorphic to the symmetric group $S_m$. Note that $p || G$ and $p \nmid |\text{Sym}(X)| = m!$. Thus $\psi$ can not be injective (otherwise, the image of $\psi$ would be a subgroup of $\text{Sym}(X)$ of order divisible by $p$, which is not possible). Thus ker $\psi$ is a non-trivial normal subgroup of $G$. Since we have seen in Problem 2 of homework 40 that ker $\psi \subseteq H$, ker $\psi$ is also a proper subgroup. Thus $G$ is not simple.

b) We keep the notation introduced in a), so now $m = p$. Let $K = \ker \psi$. By the First Isomorphism Theorem, the image of $\psi$ is isomorphic to $G/K$ and it is a subgroup of $\text{Sym}(X)$, so its order divides $p!$. It also divides $|G|$, so $|G/K|$ divides $\gcd(|G|, p!)$. Since all prime divisors of $|G|$ are larger or equal to $p$, we have $\gcd(|G|/p, (p−1)!) = 1$ so $\gcd(|G|, p!) = p$. Thus $|G/K|$ divides $p$, so it is equal to $p$. Thus $[G : K] = p = [G : H]$. Since $K \subseteq H$, we must have $H = K$. Thus $H$ is normal in $G$.

**Problem 3.** Let $G$ be a finite group and let $P$ be a Sylow $p$–subgroup of $G$ (for some prime $p$).

a) Suppose that $f : G \longrightarrow H$ is a surjective homomorphism. Prove that $f(P)$ is a Sylow $p$–subgroup of $H$.

b) Let $N$ be a normal subgroup of $G$. Prove that $P \cap N$ is a Sylow $p$–subgroup of $N$.

**Solution:** a) Let $|G| = p^a m$, $p \nmid m$, $K = \ker f$ and $|K| = p^b n$, $p \nmid n$. Since $|K||G|$, we have $b \leq a$ and $n|m$. By the First Isomorphism Theorem, $H$ is isomorphic to $G/K$, so $|H| = p^{a-b} m/n$. Again by the First Isomorphism Theorem, $f(P)$ is isomorphic to $P/(P \cap K)$. Since $P \cap K$ is a $p$–subgroup of $K$, $|P \cap K| = p^c$ for some $c \leq b$. Thus $|f(P)| = p^{a-c}$. On the other hand, $p^{a-c} = |f(P)||H|$, so $a - c \leq a - b$, so
i.e. $c \geq b$. Thus $c = b$ and $|f(P)| = p^{a-b}$, i.e. $f(P)$ is a Sylow $p$–subgroup of $H$.

b) Apply the solution to a) to the canonical homomorphism $f : G \rightarrow G/N$, so $N = \ker f = K$. We have seen that $|P \cap K| = p^b$, where $|K| = p^bn$, $p \nmid n$. Thus $P \cap K = P \cap N$ is a Sylow $p$–subgroup of $N$.

**Remark.** Look also at solution to Problem 1 of Homework 37.