Homework
due on Friday, December 7

Read sections 2.10 in Lauritzen's book and sections 7.1.1, 7.1.2 in Cameron's book.
Solve the following problems.

Problem 1. Let p be a prime. Prove that every group of order p^2 is abelian. Hint: Use problem 2 of homework 35.

Problem 2. Let G be a finite group of order pq, where $p < q$ are primes.

a) Show that G has a normal Sylow $q-$subgroup.

b) Suppose that $p \nmid (q-1)$. Prove that G has a normal Sylow $p-$subgroup.

c) Suppose that $p \nmid (q-1)$. Let P be the Sylow $p-$subgroup of G and let Q be the Sylow $q-$subgroup of G. Prove that elements of P commute with elements of Q (problem 3b) from Test III can be useful). Conclude that G is a cyclic group.

Problem 3. This problem sketches a different proof of existence of Sylow $p-$subgroups. Let p be a prime. Let G be a finite group and suppose that every group of order smaller than $|G|$ has a Sylow $p-$subgroup (so this proof goes by induction on $|G|$). If $p \nmid |G|$, there is nothing to prove, so we assume that $p||G|$. We use the action of G on itself by conjugation. Recall that the stabilizer of an elementa $a \in G$ is simply its centralizer $C(a)$ (and orbits are the conjugacy classes). In particular, the fixed points of this action are the elements of the center $Z(G)$.

a) Prove that if $p \nmid |Z(G)|$ then there is a non-central element a whose conjugacy class has size not divisible by p. Then justify the following claims:

- $C(a)$ is a proper subgroup of G so it has a Sylow $p-$subgroup P;
- the index $[G : C(a)]$ is prime to p;
- P is a Sylow P subgroup of G.

b) Suppose that $p||Z(G)|$ and that $Z(G)$ has an element g of order p. Show that $Q =< g >$ is a normal subgroup of G of order p. Conisder the canonical homomorphism $f : G \longrightarrow G/Q$. Since $|G/Q| < |G|$, G/Q has a Sylow $p-$subgroup P. Prove that $f^{-1}(P)$ is a Sylow $p-$subgroup of G.

c) Suppose that $p||Z(G)|$ and $Z(G)$ has no elements of order p (we know that this is not possible by Cauchy's Theorem, but I do not want to use this theorem, since we proved it using Sylow Theorem). Let $a \in Z(G)$ be a non-trivial element. Show that $p \nmid | < a > |$. Show that $Z(G)/ < a >$ has no elements of order p. Since $|Z(G)/ < a > | < |G|$, $Z(G)/ < a >$ has a Sylow $p-$subgroup P. Show that P is non-trivial and has an element of order p, a contradiction.