Solution to Problems 18: We are asked to solve the system

$$x \equiv 17 \pmod{504}\ ,\ x \equiv -4 \pmod{35}\ ,\ x \equiv 33 \pmod{16}\ .$$

Since the moduli are not pairwise relatively prime we can not apply Chinese remainder theorem to this system. Note that $504 = 8 \cdot 9 \cdot 7$ and $35 = 5 \cdot 7$. Now the congruence $x \equiv 17 \pmod{504}$ is equivalent to the three congruences

$$x \equiv 17 \equiv 1 \pmod{8}\ ,\ x \equiv 17 \equiv -1 \pmod{9}\ ,\ x \equiv 17 \equiv 3 \pmod{7}\ .$$

Similarly, the congruence $x \equiv -4 \pmod{35}$ is equivalent to the system

$$x \equiv -4 \equiv 1 \pmod{5}\ ,\ x \equiv 3 \pmod{7}\ .$$

Finally the congruence $x \equiv 33 \pmod{16}$ is the same as $x \equiv 1 \pmod{16}$. So our original system is equivalent to the system

$$x \equiv 1 \pmod{16}\ ,\ x \equiv -1 \pmod{9}\ ,\ x \equiv 3 \pmod{7}\ ,\ x \equiv 1 \pmod{5}\ .$$

(note that we did not include $x \equiv 1 \pmod{8}$ since this congruence is a consequence of $x \equiv 1 \pmod{16}$).

Now we may apply Chinese remainder theorem. We calculate that

$$(-59) \cdot 16 + 3 \cdot (9 \cdot 7 \cdot 5) = 1,$$

$$249 \cdot 9 + (-4) \cdot 16 \cdot 7 \cdot 5 = 1,$$

$$103 \cdot 7 + (-1) \cdot 16 \cdot 9 \cdot 5 = 1,$$

$$(-403) \cdot 5 + 2 \cdot (16 \cdot 9 \cdot 7) = 1.$$

A solution to our system is therefore given by

$$x = 3 \cdot (9 \cdot 7 \cdot 5) + (-1) \cdot (-4) \cdot 16 \cdot 7 \cdot 5 + 3 \cdot (-1) \cdot 16 \cdot 9 \cdot 5 + 2 \cdot (16 \cdot 9 \cdot 7) = 3041.$$

All solotions to this sustem are theorefore given by

$$x = 3041 + k \cdot 16 \cdot 9 \cdot 7 \cdot 5 = 3041 + k \cdot 5040,$$

where $k \in \mathbb{Z}$.

Solution to Problems 20: The problem asks us to find smalest positive solution to the system of congruences

$$x \equiv 1 \pmod 2\ ,\ x \equiv 1 \pmod 3\ ,\ x \equiv 1 \pmod 4\ ,$$

$$x \equiv 1 \pmod 5\ ,\ x \equiv 1 \pmod 6\ ,\ x \equiv 0 \pmod 7\ .$$

The moduli are not pairwise relatively prime here. Note however that the congruences

$$x \equiv 1 \pmod 3\ ,\ x \equiv 1 \pmod 4\ ,\ x \equiv 1 \pmod 5\ ,\ x \equiv 0 \pmod 7$$

imply the remaining two congruences. So the latter systm is equivalent to the former. Now the moduli are pairwise relatively prime and we can apply Chinese remainder theorem.

$$47 \cdot 3 + (-1) \cdot 4 \cdot 5 \cdot 7 = 1,$$

$$(-26) \cdot 4 + 3 \cdot 5 \cdot 7 = 1,$$

$$17 \cdot 5 + (-1) \cdot \cdot 3 \cdot 4 \cdot 7 = 1,$$

$$(-17) \cdot 7 + 2 \cdot 3 \cdot 4 \cdot 5 = 1.$$

A solution to our system is given by

$$x = (-1) \cdot 4 \cdot 5 \cdot 7 + 3 \cdot 5 \cdot 7 + (-1) \cdot \cdot 3 \cdot 4 \cdot 7 + 0 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = -119.$$

The smallest positive solution is then $-119 + 3 \cdot 4 \cdot 5 \cdot 7 = 301$.