Problem 1. Let \( p, q \) be distinct prime numbers. Prove that

\[
p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}.
\]

Solution: Since \( p \neq q \) are prime numbers, we have \( \gcd(p, q) = 1 \). By Fermat’s Little Theorem, \( p^{q-1} \equiv 1 \pmod{q} \). Clearly \( q^{p-1} \equiv 0 \pmod{q} \). Thus

\[
p^{q-1} + q^{p-1} \equiv 1 \pmod{q}.
\]

Exchanging the roles of \( p \) and \( q \) in the above argument, we prove that

\[
p^{q-1} + q^{p-1} \equiv 1 \pmod{p}.
\]

In other words, \( p^{q-1} + q^{p-1} - 1 \) is divisible by both \( p \) and \( q \). Since \( p \) and \( q \) are relatively prime, we conclude that \( p^{q-1} + q^{p-1} - 1 \) is divisible by \( pq \), i.e. \( p^{q-1} + q^{p-1} \equiv 1 \pmod{pq} \).

Problem 2. Let \( m, n \) be positive integers such that \( m \mid n \). Prove that \( \phi(m) \mid \phi(n) \) and that \( \phi(mn) = m\phi(n) \)

Solution: Since \( m \mid n \), we can number the prime divisors of \( n \) such that

\[m = p_1^{a_1} \cdots p_s^{a_s} \quad \text{and} \quad n = p_1^{b_1} \cdots p_s^{b_s} p_{s+1}^{b_{s+1}} \cdots p_t^{b_t},\]

where \( t \geq s \), \( 0 < a_i \leq b_i \) for \( i = 1, 2, \ldots, s \) and \( 0 < b_i \) for \( i > s \), and \( p_1, \ldots, p_t \) are pairwise distinct prime numbers.

Now

\[
\phi(m) = (p_1 - 1)p_1^{a_1 - 1}(p_2 - 1)p_2^{a_2 - 1}\cdots(p_s - 1)p_s^{a_s - 1}
\]

and

\[
\phi(n) = (p_1 - 1)p_1^{b_1 - 1}\cdots(p_s - 1)p_s^{b_s - 1}(p_{s+1} - 1)p_{s+1}^{b_{s+1} - 1}\cdots(p_t - 1)p_t^{b_t - 1}.
\]

It is clear now that \( \phi(m) \mid \phi(n) \). Moreover, \( mn = p_1^{a_1+b_1} \cdots p_s^{a_s+b_s} p_{s+1}^{b_{s+1}} \cdots p_t^{b_t} \) and

\[
\phi(mn) = (p_1 - 1)p_1^{a_1+b_1 - 1}\cdots(p_s - 1)p_s^{a_s+b_s - 1}(p_{s+1} - 1)p_{s+1}^{b_{s+1} - 1}\cdots(p_t - 1)p_t^{b_t - 1} = m\phi(n).
\]

Second solution: Suppose that the result is false and let \( m \mid n \) be a counterexample with smallest possible \( n \). Clearly \( m > 1 \) (since the result holds trivially for \( m = 1 \)). Let \( p \) be a prime divisor of \( m \). Thus we can write \( m = p^a m_1 \) and \( n = p^b n_1 \) for some
0 < a \leq b \text{ and natural numbers } n_1, m_1 \text{ not divisible by } p. \text{ Since } m_1 \mid n = p^b n_1 \text{ and } \gcd(p, m_1) = 1, \text{ we have } m_1 \mid n_1. \text{ Also } \phi(m) = \phi(p^a)\phi(m_1) = (p - 1)p^{a-1}\phi(m_1), \phi(n) = \phi(p^b)\phi(n_1) = (p - 1)p^{b-1}\phi(n_1) \text{ and } \phi(mn) = \phi(p^{a+b})\phi(m_1n_1) = (p - 1)p^{a+b-1}\phi(m_1n_1). \text{ Since } m_1 \mid n_1 \text{ and } n_1 < n, \text{ the result is true for } m_1, n_1, \text{ i.e. } \phi(m_1)\phi(n_1) \text{ and } \phi(m_1n_1) = m_1\phi(n_1). \text{ But then } \phi(m) = (p - 1)p^{a-1}\phi(m_1)|(p - 1)p^{b-1}\phi(m_1)|(p - 1)p^{b-1}\phi(n_1) = \phi(n) \text{ and } \phi(mn) = (p - 1)p^{a+b-1}\phi(m_1n_1) = p^a m_1(p - 1)p^b\phi(n_1) = m\phi(n) \text{ so the result is true for } m, n \text{ contrary to our assumption. The contradiction proves that no counterexample to our result exists.}

**Problem 3.** Compute \( \phi(2592) \), \( \phi(111111) \), \( \phi(15!) \).

**Solution:** We have

\[
2592 = 4 \cdot 648 = 4 \cdot 4 \cdot 162 = 2^5 \cdot 81 = 2^5 \cdot 3^4
\]

Thus \( \phi(2592) = \phi(2^5)\phi(3^4) = 2^4 \cdot 2 \cdot 3^3 = 2^5 \cdot 3^3 \).

Clearly 111111 is divisible by 11,3 so

\[
111111 = 11 \cdot 10101 = 11 \cdot 3 \cdot 3367
\]

Now 3367 is divisible by 7: 3367 = 7 \cdot 481. The next prime to consider is 13 and indeed 481 = 13 \cdot 37. Thus 111111 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 and

\[
\phi(111111) = \phi(3)\phi(7)\phi(11)\phi(13)\phi(37) = 2 \cdot 6 \cdot 10 \cdot 12 \cdot 36 = 2^7 \cdot 3^4 \cdot 5.
\]

Finally 15! = \( 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \), so

\[
\phi(15!) = \phi(2^{11})\phi(3^6)\phi(5^3)\phi(7^2)\phi(11)\phi(13) = 2^{10} \cdot 2 \cdot 3^5 \cdot 4 \cdot 5^2 \cdot 6 \cdot 7 \cdot 10 \cdot 12 = 2^{17} \cdot 3^7 \cdot 5^3 \cdot 7.
\]
Problem 4. Prove that 561 is a composite number and \( a^{561} \equiv a \pmod{561} \) for every integer \( a \).

Solution: Let us first note the following corollary from Fermat’s Little Theorem:

Proposition 1. Let \( p \) be a prime number. For any integer \( n \) and any natural number \( k \) we have \( n^{k(p-1)+1} \equiv n \pmod{p} \).

Indeed, if \( p|a \) then both sides of the congruence are \( \equiv 0 \pmod{p} \) and if \( \gcd(p,a) = 1 \) then \( n^{p-1} \equiv 1 \pmod{p} \) and \( n^{k(p-1)+1} = n(n^{p-1})^k \equiv n \pmod{p} \).

We have 561 = 3 \cdot 187 = 3 \cdot 11 \cdot 17, so 561 is not a prime. Now 561 = 280 \cdot 2 + 1 = 56 \cdot 10 + 1 = 35 \cdot 16 + 1. By the proposition, \( n^{561} \equiv n \pmod{3} \), \( n^{561} \equiv n \pmod{11} \) and \( n^{561} \equiv n \pmod{17} \) for every integer \( n \). Thus \( n^{561} - n \) is divisible by 3, 11, 17 and since these numbers are pairwise relatively prime, \( 3 \cdot 11 \cdot 17 | n^{561} - n \), i.e. \( n^{561} \equiv n \pmod{561} \) for every integer \( n \).