Homework
due on Tuesday, February 26

Problem 1. Let K be a field of characteristic $p > 0$. For $a \in K$ define $f_a(x) = x^p - x - a$.

a) Prove that if u is a root of f_a in some field L containing K then $u + 1$ is also a root of f_a. Conclude that if f_a has a root in L then f_a splits into linear factors in L.

b) Use a) to show that if u, w are roots of f_a in some field extension L of K then $K(u) = K(w)$. Conclude that all irreducible factors of f_a in $K[x]$ must have the same degree.

c) Prove that if f_a does not have a root in K then f_a is irreducible in $K[x]$. Conlude that for any non-zero $a \in \mathbb{F}_p$ the polynomial $x^p - x - a$ is irreducible in $\mathbb{F}_p[x]$.

Problem 2. Let $\Phi_n(x)$ be the $n-$th cyclotomic polynomial.

a) Prove that if $n > 1$ is odd then $\Phi_{2n}(x) = \Phi_n(-x)$.

b) Prove that $\Phi_n(1) = p$ if n is a power of a prime p and $\Phi_n(1) = 1$ for all other n.

Problem 3. Let K be a field. Suppose that K contains a primitive n-th root of 1 and a primitive m-th root of 1. Prove that K contains a primitive N-th root of 1, where $N = \operatorname{lcm}(m, n)$.

Problem 4. Let $\mathbb{Q}(\zeta_n)$ be the $n-$th cyclotomic field.

a) Prove that $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_n)$ for $m \leq n$ iff either $m = n$ or m is odd and $n = 2m$.
Hint. Show that if $a|b$ and $\phi(a) = \phi(b)$ then either $a = b$ or a is odd and $b = 2a$.

b) Prove that $\mathbb{Q}(\zeta_m) \subseteq \mathbb{Q}(\zeta_n)$ iff either $m|n$ or n is odd and $m|2n$..

c) Prove that the composite $\mathbb{Q}(\zeta_m)\mathbb{Q}(\zeta_n)$ is equal to $\mathbb{Q}(\zeta_N)$ where $N = \operatorname{lcm}(m, n)$.

Remark. It is true that $\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta_d)$, where $d = \gcd(m, n)$, but it is a bit harder to prove.