Homework

due on Tuesday, March 13

Read sections 14.1, 14.2, 14.3, 14.4, 14.5 in the book (especilly the examples and statements of results; in class we had different proofs of some the main theorems).

Problem 1. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Prove that $F/\mathbb{Q}$ is Galois with Galois group G isomorphic to the direct product of two cyclic groups of order 2. List the elements of G by showing how they act on $\sqrt{2}$ and $\sqrt{3}$.

Problem 2. Let $K = \mathbb{Q}(\sqrt{2})$ and $L = \mathbb{Q}(\sqrt[4]{2})$. Show that $K/\mathbb{Q}$ and L/K are normal but $L/\mathbb{Q}$ is not normal.

Problem 3. Let p be an odd prime. Prove that the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$ contains unique subfield K of degree 2 over $\mathbb{Q}$ (i.e. quadratic subfield). In other words, there is unique square-free integer m such that $\sqrt{m} \in \mathbb{Q}(\zeta_p)$.

Challenge: What is m?

Problem 4. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Let $a = (2 + \sqrt{2})(3 + \sqrt{3})$.

a) Prove that a is not a square in F. Hint: Asume that $a = c^2$ is a square in F. Let τ be the non-trivial element of $Gal(F/\mathbb{Q}(\sqrt{2}))$. Show that $x\tau(x) \in \mathbb{Q}(\sqrt{2})$ for every $x \in F$. Conclude that $a\tau(a)$ is a square in $\mathbb{Q}(\sqrt{2})$. Show that $a\tau(a) = 6(2 + \sqrt{2})^2$ and conclude that 6 is a square in $\mathbb{Q}(\sqrt{2})$. Derive a contradiction.

b) Let $\alpha = \sqrt{a}$ and $E = \mathbb{Q}(\alpha)$. Show that $[E : \mathbb{Q}] = 8$ and that the roots of the minimal polynomial of α over $\mathbb{Q}$ are the 8 elements $\pm\sqrt{(2 \pm \sqrt{2})(3 \pm \sqrt{3})}$.

c) Show that all 8 roots of the minimal polynomial of α are in E and conclude that $E/\mathbb{Q}$ is Galois. Show that if β is any of these eight roots then there is unique automorphism τ of E such that $\tau(\alpha) = \beta$.

d) Let σ be the automorphism of E which maps α to $\gamma = \sqrt{(2 - \sqrt{2})(3 + \sqrt{3})}$. Show that $\sigma(\sqrt{2}) = -\sqrt{2}$ and $\sigma(\sqrt{3}) = \sqrt{3}$. Conclude thta $\sigma(\alpha\gamma) = -\alpha\gamma$ and $\sigma(\gamma) = -\alpha$. Finally conclude that σ has order 4.

e) As in d), show that the automorphism ρ of E which maps α to $\sqrt{(2 + \sqrt{2})(3 - \sqrt{3})}$ has order 4. Show that $\rho^2 = \sigma^2$ and $\sigma\rho = \rho\sigma^3$. Conclude that $Gal(E/\mathbb{Q})$ is isomorphic to the quaternion group (see appropriate homework from Fall to refresh your knowledge of the quaternion group).