Homework

due on Tuesday, April 15

Problem 1. Let G be a subgroup of S_n (which we think of as permutations of $\{1, 2, ..., n\}$) such that G contains a transposition, an $(n-1)$-cycle and it acts transitively on $\{1, 2, ..., n\}$ (which means that for any i, j there is an element $\pi \in G$ such that $\pi(i) = j$). Prove that $G = S_n$.

Problem 2. Let $K \subseteq F$ be fields. Suppose that L, M are subfields of F containing K such that L/K is finite and Galois.

a) Prove that LM/M is finite and Galois.

b) Show that every automorphism of LM/M restricts to an automorphism of L/K and this restriction defines an injective homomorphism $\phi : Gal(LM/M) \longrightarrow Gal(L/K)$.

c) Show that the fixed field of the image of ϕ is the field $M \cap L$. Conclude that the image of ϕ coincides with $Gal(L/M \cap L)$.

d) Use c) to show that $[LM : M] = [L : L \cap M]$.

e) Prove that if $[M : K]$ is finite then

$$[LM : K][L \cap M : K] = [L : K][M : K].$$

Problem 3. Let G be a group. Define the **derived** subgroup $[G, G]$ of G (often called the **commutator** subgroup) as the subgroup generated by all elements of the form $a^{-1}b^{-1}ab$, where $a, b \in G$ (such elements are called **commutators**).

a) Prove that $[G, G]$ is a characteristic subgroup of G, i.e. if $f : G \longrightarrow G$ is an automorphism of G then $f([G, G]) = [G, G]$. Conclude that $[G, G]$ is normal in G (a characteristic subgroup must be normal).

b) Let N be a normal subgroup of G. Prove that G/N is abelian iff $[G, G] \subseteq N$.

c) Prove that if G is solvable then $[G, G] \neq G$ (groups for which $G = [G, G]$ are called **perfect**).

d) Define the **derived series** of G inductively as follows: $G_0 = G$, $G_{n+1} = [G_n, G_n]$. Prove that each G_n is a characteristic subgroup of G.

e) Prove that G is solvable iff $G_n = 1$ for some n.

f) Suppose that G is solvable and $G = H_0 \supseteq H_1 \supseteq ... \supseteq H_m = 1$ is a chain of subgroups such that H_{i+1} is normal in H_i and H_i/H_{i+1} is abelian for all i. Show that $G_i \subseteq H_i$ for all i. Conclude that the derived series in a solvable group is a shortest chain of subgroups with abelian succesive quotients which terminates at the trivial group. The first n such that $G_n = 1$ is called the **solvability class** of G.