Homework

due on Tuesday, May 6

Problem 1. Recall that for any sequence $\mathbf{a} = (a_1, ..., a_k)$ of non-negative real numbers we define a term ordering $\leq_{\mathbf{a}}$ on $\mathbb{N}^k$ as follows: $(m_1, ..., m_k) \leq_{\mathbf{a}} (n_1, ..., n_k)$ iff either $a_1m_1+a_2m_2+...+a_km_k < a_1n_1+a_2n_2+...+a_kn_k$ or $a_1m_1+a_2m_2+...+a_km_k = a_1n_1 + a_2n_2 + ... + a_kn_k$ and $(m_1, ..., m_k) \leq_{lex} (n_1, ..., n_k)$. Prove that if $\mathbf{a}$ and $\mathbf{b}$ are linearly independent vectors with non-negative coordinates then the corresponding orders $\leq_{\mathbf{a}}$, $\leq_{\mathbf{b}}$ are different. (**Hint.** Note that for any positive real number c the vectors $\mathbf{a}$ and $c\mathbf{a}$ define the same orders. It follows that if a_1 and b_1 are not zero then we may assume that $a_1 = b_1 = 1$. There is $i > 1$ such that $a_i \neq b_i$ and we may assume that $a_i > b_i$. Show that there are natural numbers m, n such that $a_im > n > b_im$ and use it to show that the orders $\leq_{\mathbf{a}}$, $\leq_{\mathbf{b}}$ are different. Then consider the cases when one of a_1, b_1 is 0.)

Problem 2. We proved that every symmetric polynomial can be expressed as a polynomial in elementary symmetric polynomials and our proof was constructive, i.e. it provides an algorithm to do so. Use this algorithm to express the polynomial $X_1^3 + X_2^3 + X_3^3$ as a polynomial in the elementary symmetric polynomials s_1, s_2, s_3. See exercise 38 to section 14.6 in Dummit and Foote.

Problem 3. Solve problem 22 to section 14.6 in Dummit and Foote.

Problem 4. Let I be an ideal of $F[X_1, ..., X_k]$. Fix a term ordering $\leq$ and let $f_1, ..., f_n$ and $g_1, ..., g_m$ be two Grobner bases for I with respect to $\leq$. Let $f \in F[X_1, ..., X_k]$ and let r_1, r_2 be the remainders of f with sepect to $f_1, ..., f_n$ and $g_1, ..., g_m$ respectively. Prove that $r_1 = r_2$.

Problem 5. Solve problems 24 and 26 to chapter 5 in Lauritzen's book..