Discovering determinants

The row-reduction technique provides us with an algorithm to compute the inverse of an invertible matrix A. Unfortunately, this algorithm does not provide any insight on how the entries of A^{-1} depend on the entries of A. It is our goal now to understand how A^{-1} is built from the entries of A. This will lead us to a discovery of a very important concept, namely the **determinant** of a square matrix.

Let us start with a 2×2 matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We assume that $a \neq 0$. In order to compute A^{-1} we employ the row-reduction technique. Thus we start with the matrix

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix}$$

and perform the following elementary row operations: $E_{2,1}(-c/a)$, $S_2(a)$ to get

$$\begin{pmatrix} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{pmatrix}$$

We see that A is invertible iff $ad - bc \neq 0$ and if this is the case then we may perform further operations: $S_2(1/(ad - bc))$, $E_{1,2}(-b)$, $S_1(1/a)$ to get

$$\begin{pmatrix} 1 & 0 & d/(ad - bc) & -b/(ad - bc) \\ 0 & 1 & -c/(ad - bc) & a/(ad - bc) \end{pmatrix}.$$

Thus $A^{-1} = 1/(ad - bc)\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. This formula was obtained under the assumption that $a \neq 0$, but it is straightforward to see that it works as well for $a = 0$. Set $D_2(A) = ad - bc$ and for any 1×1 matrix $B = (b)$ define $D_1(B) = b$. Hence we proved that a 2×2 matrix A is invertible iff $D_2(A) \neq 0$ and then

$$A^{-1} = \frac{1}{D_2(A)} \begin{pmatrix} D_1((d)) & -D_1((b)) \\ -D_1((c)) & D_1((a)) \end{pmatrix}$$

The reason why we wrote the inverse of A in such a strange form using D_1 will become clear soon.

Now we try to perform similar computations for a 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We assume that $a \neq 0$ and consider the matrix

$$\begin{pmatrix} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \\ g & h & i & 0 & 0 & 1 \end{pmatrix}$$

We perform on this matrix elementary operations $E_{2,1}(-d/a)$, $E_{3,1}(-g/a)$, $S_2(a)$, $S_3(a)$ and get

$$\begin{pmatrix} a & b & c & 1 & 0 & 0 \\ 0 & ea-bd & fa-dc & -d & a & 0 \\ 0 & ha-gb & ia-gc & -g & 0 & a \end{pmatrix}$$

We see that A is invertible iff the matrix $\left(\begin{smallmatrix} ea-bd & fa-dc \\ ha-gb & ia-gc \end{smallmatrix}\right)$ is invertible (i.e. has rank 2) and we already know that this is equivalent to $(ea-bd)(ia-gc)-(fa-dc)(ha-gb) \neq 0$, i.e. to $a(aei+bfg+cdh-ceg-afh-bdi) \neq 0$. Since we have assumed that $a \neq 0$, A is invertible iff $aei+bfg+cdh-ceg-afh-bdi \neq 0$. Set $D = aei+bfg+cdh-ceg-afh-bdi$. Now we could perform further elementary row operations, but we will be more clever here and note that the product $S_3(a)S_2(a)E_{3,1}(-g/a)E_{2,1}(-d/a)$ of elementary matrices representing row operations performed so far equals

$$\begin{pmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{pmatrix}$$

This means that

$$\begin{pmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ 0 & ea-bd & fa-dc \\ 0 & ha-gb & ia-gc \end{pmatrix}.$$

which is equivalent to

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \begin{pmatrix} a & b & c \\ 0 & ea-bd & fa-dc \\ 0 & ha-gb & ia-gc \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{pmatrix}$$

It is quite easy to observe that the inverse of a matrix of the form

$$M = \begin{pmatrix} a & b & c \\ 0 & x & y \\ 0 & z & t \end{pmatrix}$$

must also be of this form too, i.e. it is

$$M^{-1} = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & u & v \\ 0 & w & s \end{pmatrix}$$

The equality $MM^{-1} = I$ implies now that $\left(\begin{smallmatrix} u & v \\ w & s \end{smallmatrix}\right)$ is the inverse to $\left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ and $\alpha = 1/a$, $\beta = -(bu + cw)/a$, $\gamma = -(bv + cs)/a$. Since we already know how to invert a 2×2 matrix, we apply this to find that

$$\begin{pmatrix} a & b & c \\ 0 & ea - bd & fa - dc \\ 0 & ha - gb & ia - gc \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & (ch - bi)/aD & (bf - ce)/aD \\ 0 & (ia - gc)/aD & (-fa + dc)/aD \\ 0 & (-ha + gb)/aD & (ae - bd)/aD \end{pmatrix}$$

Thus

$$A^{-1} = \begin{pmatrix} 1/a & (ch - bi)/aD & (bf - ce)/aD \\ 0 & (ia - gc)/aD & (-fa + dc)/aD \\ 0 & (-ha + gb)/aD & (ae - bd)/aD \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -d & a & 0 \\ -g & 0 & a \end{pmatrix}$$

$$= \frac{1}{D} \begin{pmatrix} (ei - fh) & -(bi - ch) & (bf - ce) \\ -(di - fg) & (ai - cg) & -(af - cd) \\ (dh - eg) & -(ah - bg) & (ae - bd) \end{pmatrix}$$

Set $D_3(A) = D = aei + bfg + cdh - ceg - afh - bdi$. Then the last formula takes the form

$$A^{-1} = \frac{1}{D_3(A)} \begin{pmatrix} D_2(\left(\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right)) & -D_2(\left(\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right)) & D_2(\left(\begin{smallmatrix} b & c \\ e & f \end{smallmatrix}\right)) \\ -D_2(\left(\begin{smallmatrix} d & f \\ g & i \end{smallmatrix}\right)) & D_2(\left(\begin{smallmatrix} a & c \\ g & i \end{smallmatrix}\right)) & -D_2(\left(\begin{smallmatrix} a & c \\ d & f \end{smallmatrix}\right)) \\ D_2(\left(\begin{smallmatrix} d & e \\ g & h \end{smallmatrix}\right)) & -D_2(\left(\begin{smallmatrix} a & b \\ g & h \end{smallmatrix}\right)) & D_2(\left(\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right)) \end{pmatrix}.$$

We proved this formula under the assumption that $a \neq 0$ but it is not hard to see that it is still true for $a = 0$. Thus A is invertible iff $D_3(A) \neq 0$ and then the inverse of A is given by the above formula.

It is now time to analyze our results and try to find a pattern in our formulas for A^{-1}. What is the (i, j) entry of the matrix on the right in our formulas? Looking at the formulas for a few minutes provides the answer: the (i, j) entry equals ± 1 times the appropriate D applied to a matrix obtained from A by removing its $j-$th row and $i-$th column. Also, the sign ± 1 is not that hard to understand: it seems equal to $(-1)^{i+j}$. This suggest the following definition:

Definition 1. *For any $n \times n$ matrix A define $A_{i,j}$ to be the matrix obtained from A by removing its $i-th$ row and $j-th$ column.*

Our computations lead us to a prediction that for each n there should be a function D_n which to each $n \times n$ matrix A assigns a scalar $D_n(A)$ (and this function is a polynomial in the entries of A) such that A is invertible iff $D_n(A) \neq 0$. Moreover, the inverse of A should be equal to $D_n(A)^{-1}A^D$ where $A^D = (d_{i,j})$ is an $n \times n$ matrix such that

$$d_{i,j} = (-1)^{i+j} D_{n-1}(A_{j,i}).$$

We have already seen that this is true for $n = 2, 3$. This also makes a prediction for a 4×4 matrices, since in order to compute A^D for such matrices we just need to know D_3. So for a matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{4,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

we compute A^D using our formula for D_3 and then multiply A and A^D. We do not provide the actual computations here, but we encourage the reader to perform such computations (it may take some time and one has to be very careful not to make any mistakes). As expected, we see that $AA^D = D_4(A)I$, where

$$\begin{aligned} D_4(A) = {} & a_{1,1}a_{2,2}a_{3,3}a_{4,4} + a_{1,2}a_{2,1}a_{3,4}a_{4,3} + a_{1,3}a_{2,4}a_{3,1}a_{4,2} + a_{1,4}a_{2,3}a_{3,2}a_{4,1} + \\ & a_{1,2}a_{2,3}a_{3,1}a_{4,4} + a_{1,2}a_{2,4}a_{3,3}a_{4,1} + a_{1,3}a_{2,2}a_{3,4}a_{4,1} + a_{1,1}a_{2,3}a_{3,4}a_{4,2} + \\ & a_{1,3}a_{2,1}a_{3,2}a_{4,4} + a_{1,4}a_{2,1}a_{3,3}a_{4,2} + a_{1,4}a_{2,2}a_{3,1}a_{4,3} + a_{1,1}a_{2,4}a_{3,2}a_{4,3} - \\ & a_{1,2}a_{2,1}a_{3,3}a_{4,4} - a_{1,3}a_{2,2}a_{3,1}a_{4,4} - a_{1,4}a_{2,2}a_{3,3}a_{4,1} - a_{1,1}a_{2,3}a_{3,2}a_{4,4} - \\ & a_{1,1}a_{2,4}a_{3,3}a_{4,2} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,2}a_{2,3}a_{3,4}a_{4,1} - a_{1,2}a_{2,4}a_{3,1}a_{4,3} - \\ & a_{1,3}a_{2,4}a_{3,2}a_{4,1} - a_{1,3}a_{2,1}a_{3,2}a_{4,2} - a_{1,4}a_{2,3}a_{3,1}a_{4,2} - a_{1,4}a_{2,1}a_{3,2}a_{4,3} \end{aligned}$$

These computations imply that if $D_4(A) \neq 0$ then A is invertible and our predicted formula for A^{-1} holds. It is not clear at this point that if $D_4(A) = 0$ then A is not invertible (theoretically, it could happen that A^D is the zero matrix), but this can be done with some extra work. Having D_4 we can in a similar way compute D_5 and so on and verify that we get formulas for the inverse of A as predicted. This gives a strong evidence that our prediction is true and even tells us an inductive method to construct D_n. In fact, let A be an $n \times n$ matrix. From our prediction that $A^{-1} = D_n(A)A^D$ we see that $AA^D = D_n(A)I$. The $(1,1)$ entry of AA^D is given by $a_{1,1}d_{1,1} + a_{1,2}d_{2,1} + ... + a_{1,n}d_{n,1} = a_{1,1}D_{n-1}(A_{1,1}) - a_{1,2}D_{n-1}(A_{1,2}) + ... + (-1)^{n+1}a_{1,n}D_{n-1}(A_{1,n})$. Since the $(1,1)$ entry of $D_n(A)I$ is $D_n(A)$, we see that there is no choice for $D_n(A)$; if it exists, it must be given by the formula

$$D_n(A) = \sum_{j=1}^{n}(-1)^{1+j}a_{1,j}D_{n-1}(A_{1,j})$$

Starting with D_1 this provides recursive definition of the D_n's. What remains is to find a precise proof that the D_n's defined in this way in fact satisfy our prediction, i.e. that $AA^D = D_n(A)I = A^DA$ holds for all $n \times n$ matrices A and that A is invertible iff $D_n(A) \neq 0$. Note that the equality $AA^D = D_n(A)I$ is equivalent to the following identities:

$$D_n(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{i,j}D_{n-1}(A_{i,j}),\ i = 1, 2, ..., n, \quad (1)$$

and

$$0 = \sum_{j=1}^{n}(-1)^{k+j}a_{i,j}D_{n-1}(A_{k,j}) \text{ for } i \neq k. \quad (2)$$

Similarly, the equality $A^DA = D_n(A)I$ is equivalent to the following identities:

$$D_n(A) = \sum_{i=1}^{n}(-1)^{i+j}a_{i,j}D_{n-1}(A_{i,j}),\ j = 1, 2, ..., n, \quad (3)$$

and

$$0 = \sum_{i=1}^{n}(-1)^{i+k}a_{i,j}D_{n-1}(A_{i,k}) \text{ for } j \neq k. \quad (4)$$

Definition 2. *The scalar $D_n(A)$ is called the* **determinant** *of A and it is usually denoted by* $\det A$*, or sometimes by* $|A|$.

The formulas (1) and (3) are called the **Laplace expansions** of the determinant by the i-th row and by the j-th column respectively. They can be used to compute determinants in a recursive way.

Example. In order to compute $D_4(A)$ for

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 1 & 3 \end{pmatrix}$$

we can use Laplace expansion by the second column to get

$$D_4 A = -2D_3(A_{1,2}) + D_3(A_{2,2}) - 0 \cdot D_3(A_{3,2}) + 0 \cdot D_3(A_{4,2}) =$$

$$= -2D_3 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{pmatrix} + D_3 \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{pmatrix}$$

Now Laplace expansion by the first row yields

$$D_3 \begin{pmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{pmatrix} = D_2 \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} - 0 \cdot D_2 \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} + D_2 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = 9 + (1-3) = 7.$$

Similarly, we use Laplace expansion by the third column to get

$$D_3 \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \end{pmatrix} = 2 \cdot D_2 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} - 0 \cdot D_2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 3 \cdot D_2 \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} =$$

$$= 2(1-3) + 3(3-1) = 2.$$

It follows that $D_4(A) = -2 \cdot 7 + 2 = -12$.

Exercise. Compute A^D and A^{-1} for the above A.

Problem: Can you predict an explicit formula for $D_n(A)$ (above we have seen such formulas for $n \leq 4$).