Math 404, Exam I
February 26, 2004

Problem 1. State a definition of (2 points each):

a) a vector space over a field F;

b) a linearly independent subset S of a vector space;

c) a basis and dimension of a vector space;

d) span(S), where S is a subset of a vector space V;

e) the rank of a matrix;

f) a linear transformation $T : V \longrightarrow W$;

g) the kernel ker$T$ and the image ImT of a linear transformation $T : V \longrightarrow W$.

Problem 2. a) Find the reduced row-echelon form of the matrix

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

List the elementary row operations performed. What is the rank of this matrix? (5 points)

b) Find the dimension and a basis of the subspace of $\mathbb{R}^5$ spanned by the vectors $(1,1,0,-1,-1)$, $(1,0,-1,0,0)$, $(1,2,1,-2,-2)$, $(2,1,1,1,2)$, $(4,3,-1,-3,-3)$. Express all these vectors as linear combination of vectors in the constructed basis. (5 points)

Problem 3. a) Solve the system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14 \end{aligned}$$

by finding a basis of the space of solutions of the associated homogeneous system and a solution to the given system (5 points). Verify your answer by checking that the vectors really are solutions.

b) Find a system of linear homogeneous equations whose solution space is spanned by the vectors from problem 2b). Verify your answer. What is the minimal possible number of equations in such a system (5 points)?

Problem 4. Let $\mathbf{v}_1 = (1,0,-1,0)$, $\mathbf{v}_2 = (1,-1,0,0)$, $\mathbf{v}_3 = (1,0,0,-1)$. Set $V = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3,\})$. Let $\mathbf{w}_1 = (1,1,1,1)$, $\mathbf{w}_2 = (1,2,1,0)$, $\mathbf{w}_3 = (0,1,2,1)$. Set $W = \text{span}(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\})$. Find a basis of $V \cap W$. What can you say about $V + W$?. (6 points)

Problem 5. a) A liner transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfies $T(1,0,1) = (1,1,0)$ and $T(1,1,0) = (1,1,1)$. What is $T(5,3,2)$? (5 points)

b) Is there a liner transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which satisfies $T(1,1) = (1,1,0)$, $T(1,-1) = (1,1,1)$ and $T(3,1) = (1,0,0)$? (5 points)

Problem 6. Answer true or false. In each case provide an explanation (2 points each).

a) If span(S) is contained in S then S is a subspace.

b) The set of all polynomials p such that $p(1) = p(2)$ is a subspace of the space of all polynomials.

c) The function $T(a, b) = (ab, a+b)$ is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$.

d) Any subspace of $\mathbb{R}^5$ of dimension 3 is a space of solutions of a homogeneous system of 2 linear equations.

e) The space of polynomials of degree at most 4 is a sum of two subspaces of dimension 2.

f) If $T : V \longrightarrow W$ is a linear transformation and $\dim V > \dim W$ then $T(v) = 0$ for some $v \neq 0$.

Problem 7. a) Let $T : V \longrightarrow W$ is a linear transformation. Suppose that $T(v_1), T(v_2), T(v_3)$ are linearly independent in W. Prove that v_1, v_2, v_3 are linearly independent in V. (5 points)

b) Suppose that $S_1 \subseteq S_2$ are subsets of a vector space V such that S_1 spans V and S_2 is linearly independent. Prove that $S_1 = S_2$. (5 points)

c) Let U be a subspace of $\mathbb{R}^n$. Prove that $U \cap U^{\perp} = \{0\}$ and that $\mathbb{R}^n = U \oplus U^{\perp}$. (5 points)

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The following problems are optional. You may earn extra points, but work on these problem only after you are done with the other problems

Problem 8. Let $T : V \longrightarrow V$ be a linear transformation. Suppose that $v \in V$ is such that $T^3(v) = 0$ but $T^2(v) \neq 0$. Prove that $v, T(v), T^2(v)$ are linearly independent. Here T^k is the composition $T \circ T \circ ... \circ T$ of T with itself $k-$times. (7 points)

Problem 9. Let $T : V \longrightarrow V$ be a linear transformation such that T and T^2 have the same image. Prove that $V = \ker T \oplus \mathrm{Im} T$. (8 points)

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$$\begin{pmatrix} 1 & 1 & 1 & 2 & 4 \\ 1 & 0 & 2 & 1 & 3 \\ 0 & -1 & 1 & 1 & -1 \\ -1 & 0 & -2 & 1 & -3 \\ -1 & 0 & -2 & 2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & 2 & 1 & -2 & -2 \\ 2 & 1 & 1 & 1 & 2 \\ 4 & 3 & -1 & -3 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & -2 & -5/2 \\ 0 & 0 & 1 & 1 & 3/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 2 & 1 \\ -1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$