Math 404, Exam II
March 24, 2004

Problem 1. State a definition of (2 points each):

a) a $T-$invariant subspace of a vector space;

b) the matrix representation $M_{\mathbf{v}}^{\mathbf{w}}(T)$ of $T : V \longrightarrow W$ in ordered bases $\mathbf{v} = v_1, ..., v_n$ of V and $\mathbf{w} = w_1, ..., w_m$ of W;

c) the transition matrix from a basis $\mathbf{v}$ to a basis $\mathbf{v}'$ of a vector space V;

d) similar matrices;

e) the elementary $n \times n$ matrix $E_{1,2}(2)$;

Problem 2. Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Verify your answer. (6 points)

b) Express A as a product of elementary matrices (write down the actual matrices). (4 points)

Problem 3. A linear transformation $S : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is given by the matrix $A = \begin{pmatrix} 2 & 6 & 1 & 4 & 4 \\ 1 & 3 & 1 & 3 & 3 \\ 1 & 3 & 2 & 5 & 5 \\ 1 & 3 & 3 & 7 & 7 \end{pmatrix}$.

Find bases of the kernel and of the image of S. (7 points)

Problem 4. a) Find the matrix of the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad T(x, y, z) = (x + y + z, x - y + z)$$

in the basis $(1, -2, 1), (1, 2, 1), (0, 2, 1)$ of $\mathbb{R}^3$ and the basis $(1, 1), (1, -1)$ of $\mathbb{R}^2$. (7 points)

b) What is the change of basis matrix from the basis $\mathbf{b} = \{(1, 1, 1), (1, 0, 1), (0, 0, 1)\}$ to the basis $\mathbf{d} = \{(2, 1, 1), (2, 2, 1), (3, 2, 2)\}$ of $\mathbb{R}^3$?. Find the coordinates of a vector in the basis $\mathbf{d}$ if its coordinates in the basis $\mathbf{b}$ are $(0, 1, 2)$. (7 points)

Problem 5. Find a linear transformation $T : \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ such that $\ker T$ has basis $(1, 0, 2, 0, -2, 3)$, $(0, 1, -1, 0, 1, 1)$, $(0, 0, 0, 1, -2, 2)$. (7 points)

Problem 6. Answer true or false. In each case provide an explanation (2 points each).

a) Every matrix is a product of elementary matrices.

b) If $T : V \longrightarrow V$ is a linear transformation then $\ker T$ is a $T-$invariant subspace.

c) There exists a linear transformation $T : K^5 \longrightarrow K^5$ such that $\ker T = \mathrm{Im} T$.

d) The matrix $\left(\begin{smallmatrix} 1 & 1 \\ 2 & 2 \end{smallmatrix}\right)$ is a transition matrix from some basis of $\mathbb{R}^2$ to some other basis.

e) The sum of two invertible matrices is always invertible.

f) If $T : V \longrightarrow V$ is a linear transformation then $\mathrm{Im} T$ is a $T-$invariant subspace.

Problem 7. a) Let $T : V \longrightarrow V$ be a linear transformation. Suppose that $V = \mathrm{Im} T \oplus W$. Show that if W is T-invariant then $W = \ker T$. (5 points)

b) Let S, T be liner transformations $V \longrightarrow V$ such that $ST = TS$. Prove that $\ker T$ is S-invariant. (5 points)

c) Let A be a matrix of size 5×3 and let B be a matrix of size 3×5. Prove that the 5×5 matrix AB is not invertible. (Hint: Think in terms of linear transformations). (5 points)

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The following problems are optional. You may earn extra points, but work on these problem only after you are done with the other problems

Problem 8. Let $T : V \longrightarrow V$ be a linear transformation such that every one dimensional subspace of V is T-invariant. Prove that $T = aI$ for some constant a. (7 points)

Problem 9. For a square matrix A, define the **trace** $\mathrm{tr}A$ as the sum of all diagonal entries of A (so if $A = (a_{i,j})$ then $\mathrm{tr}A = a_{1,1} + a_{2,2} + a_{3,3} + \ldots$). Prove that for any two 3×3 matrices A, B we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Conclude that if M, N are similar 3×3 matrices then $\mathrm{tr}M = \mathrm{tr}N$ (all this is true for matrices of arbitrary size). Are the matrices

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}$$

similar? (8 points)

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$$\begin{pmatrix} 2 & 6 & 1 & 4 & 4 \\ 1 & 3 & 1 & 3 & 3 \\ 1 & 3 & 2 & 5 & 5 \\ 1 & 3 & 3 & 7 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 2 & 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 & 2 & 2 & 3 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{pmatrix}$$