Math 404, Exam III
April 29, 2004; Due May 5, 2004

Problem 1. Find the minimal and characteristic polynomials for the linear transformation $T : \mathbf{R}^5 \longrightarrow \mathbf{R}^5$ given by the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -3 & 2 & 2 & 0 & -2 \\ 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & -2 & 2 & 2 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Find a rational canonical decomposition for T. Compute B^{1000}.

Problem 2. Let $T : \mathbb{R}^4 \longrightarrow R^4$ be given by $T(a, b, c, d) = (a + b, b + c, c + d, d + a)$.

a) Find the annihilator of $v = (1, 0, -1, 0)$ and of $w = (1, 0, 0, 0)$.

b) Find the minimal polynomial of T.

c) Find a rational canonical form of T and a basis in which T has this form.

Problem 3. a) Find the characteristic polynomial, the eigenvalues and the eigenvectors of the matrix

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{pmatrix}$$

Is this matrix diagonalizable?

b) Find all x such that the determinant of the matrix

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & x & x \\ x & 1 & 0 & -1 \\ x & 0 & 1 & 1 \end{pmatrix}$$

equals 0.

Solve 3 of the following problems:

Problem 4. Let U be a $T-$invariant subspace of V and let S be a subset of V. Show that the set I of all polynomials f such that $f(T)(v) \in U$ for every $v \in S$ (i.e. $I = \{f : f(T)(v) \in U \text{for all} v \in S\}$) is an ideal. Show that this ideal contains non-zero polynomials. Consider the case when $U = \{0\}$ and $S = V$ and conclude that there exists a monic polynomial q_T such that for any polynomial f, we have $f(T) = 0$ iff $q_T | f$. (Of course, q_T is the minimal polynomial of T.)

Problem 5. Let $T : V \longrightarrow V$ be a linear transformation and let $v \in V$ be a non-zero vector.

a) Prove that any $T-$invariant subspace of a cyclic subspace is cyclic.
Hint. Let U be a T-invariant subspace of $< v >$. Consider the unique monic polynomial q with the property that for any polynomial f, we have $f(T)(v) \in U$ iff $q|f$. Show that $U =< w >$, where $w = q(T)(v)$. Show also that $q|p_v$.

b) Prove that if p_v is a power of an irreducible polynomial and U, W are T-invariant subspaces of $< v >$ then either $U \subseteq W$ or $W \subseteq U$. Conclude that $< v >$ cannot be decomposed into a direct sum of proper T-invariant subspaces.

Problem 6. Let $T : V \longrightarrow V$ be a linear transformation and let $v_1, ..., v_n$ be a basis of V. Prove that the minimal polynomial q_T is equal to the least common multiple of the annihilators $p_{v_1}, ..., p_{v_n}$ of $v_1, ..., v_n$.

Problem 7. a) Let $T : V \longrightarrow V$ be a linear transformation. Prove that $\ker T^i \subseteq \ker T^{i+1}$ and $\mathrm{Im} T^{i+1} \subseteq \mathrm{Im} T^i$ for every non-negative integer i. Prove furthermore that if $\ker T^k$ and $\ker T^{k+1}$ have the same dimension for some integer k then all the kernels $\ker T^i$ have the same dimension for $i \geq k$.

b) Use a) to show that if the matrices A^k and A^{k+1} have the same rank, then all the matrices A^i with $i \geq k$ have the same rank.