1 Sample problems.

I. Let $\mathbf{v}_1 = (1, 1, 2, 2, 5)$, $\mathbf{v}_2 = (-1, 1, -2, 0, -1)$, $\mathbf{v}_3 = (0, 0, 1, 0, 1)$, $\mathbf{v}_4 = (0, 1, 0, 1, 2)$, $\mathbf{v}_5 = (1, 1, 5, 2, 8)$. Set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$. Among the vectors in S find a basis of $\text{span}(S)$ which contains $\mathbf{v}_1$ and $\mathbf{v}_4$. Express the vectors in S as linear combinations of vectors in the basis found.

Solution: We arrange the vectors from S as columns of a matrix M such that $\mathbf{v}_1$ and $\mathbf{v}_4$ are the first columns of M:

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & -2 & 1 & 5 \\ 2 & 1 & 0 & 0 & 2 \\ 5 & 2 & -1 & 1 & 8 \end{pmatrix}$$

Next we find the reduced row-echelon form N of M:

$$N = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The columns of M corresponding to pivot columns of N form a required basis, i.e. $\mathbf{v}_1$, $\mathbf{v}_4$, $\mathbf{v}_3$ is a basis of $\text{span}(S)$. To express $\mathbf{v}_2$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_4$, $\mathbf{v}_3$ we look at the third column of N (which corresponds to $\mathbf{v}_2$). It tells us that

$$\mathbf{v}_2 = -\mathbf{v}_1 + 2\mathbf{v}_4.$$

Similarly, from the fifth column of N we read that

$$\mathbf{v}_5 = \mathbf{v}_1 + 3\mathbf{v}_3.$$

II. Solve the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 &= 1 \\ x_1 + x_3 + x_4 + 3x_5 + 2x_6 + 3x_7 &= 2 \\ 3x_1 + 2x_2 + 3x_3 + 2x_4 + 3x_5 + 2x_6 + 3x_7 &= 2 \\ 2x_1 - x_2 + 2x_3 - x_4 + 2x_5 - x_6 + 2x_7 &= -1 \end{aligned}$$

Solution: The augmented matrix of this system is

$$\left(\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 3 & 2 & 3 & 2 \\ 2 & -1 & 2 & -1 & 2 & -1 & 2 & -1 \end{array}\right)$$

The reduced row-echelon form of this matrix is

$$\left(\begin{array}{ccccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 & -2 & -2 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The last column is not a pivot column so the system is consistent. The free variables are x_3, x_5, x_6 and x_7 and the system takes form:

$$
\begin{aligned}
x_1 &= -x_3 - x_5 - x_7 \\
x_2 &= 2x_5 + x_6 + 2x_7 - 1 \\
x_3 &= x_3 \\
x_4 &= -2x_5 - 2x_6 - 2x_7 - 2 \\
x_5 &= x_5 \\
x_6 &= x_6 \\
x_7 &= x_7
\end{aligned}
$$

We see that a basis of solutions to the associated homogeneous system is $\mathbf{u}_1 = (-1,0,1,0,0,0,0)$, $\mathbf{u}_2 = (-1,2,0,-2,1,0,0)$, $\mathbf{u}_3 = (0,1,0,-2,0,1,0)$, $\mathbf{u}_4 = (-1,2,0,-2,0,0,1)$.

A particular solution to the non-homogeneous system is obtained by setting $x_3 = x_5 = x_6 = x_7 = 0$. We get $(0,-1,0,2,0,0,0)$.

The general solution to the system can be written as:

$$(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = (0,-1,0,2,0,0,0) + a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4$$

where a_1, a_2, a_3, a_4 are independent parameters.

III. Let $\mathbf{w}_1 = (0,1,1,0,1,1,0)$, $\mathbf{w}_2 = (1,1,1,0,-1,-1,0)$, $\mathbf{w}_3 = (2,3,3,0,-1,-1,-2)$, $\mathbf{w}_4 = (2,1,2,0,0,-1,2)$, $\mathbf{w}_5 = (1,2,2,0,0,0,1)$. Find a basis of $\text{span}(S)^\perp$, where $S = \{\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4,\mathbf{w}_5\}$.

Solution: This problem simply asks for a basis of solutions to the homogeneous system of equations whose coefficient matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & -1 & -1 & 0 \\ 2 & 3 & 3 & 0 & -1 & -1 & -2 \\ 2 & 1 & 2 & 0 & 0 & -1 & 2 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}$$

has rows equal to vectors in S. The reduced row echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that x_4, x_5 and x_6 are free variables and

$$
\begin{aligned}
x_1 &= 2x_5 + 2x_6 \\
x_2 &= 2x_5 + x_6 \\
x_3 &= -3x_5 - 2x_6 \\
x_4 &= x_4 \\
x_5 &= x_5 \\
x_6 &= x_6 \\
x_7 &= 0
\end{aligned}
$$

A basis of solutions is $(0,0,0,1,0,0,0)$, $(2,2,-3,0,1,0,0)$, $(2,1,-2,0,0,1,0)$. This is a basis of $\text{span}(S)^\perp$.

IV. Let $\mathbf{v}_1 = (1,1,1,1,1,1,1)$, $\mathbf{v}_2 = (1,0,1,1,3,2,3)$, $\mathbf{v}_3 = (3,2,3,2,3,2,3)$, $\mathbf{v}_4 = (2,-1,2,-1,2,-1,2)$. Set $T = \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$. Find a system of homogeneous equations whose solution space is $\text{span}(T)$.

Solution: First we find a basis of $\text{span}(T)^\perp$ using III. We get $(-1,0,1,0,0,0,0)$, $(-1,2,0,-2,0,1,0)$, $(0,1,0,-2,0,1,0)$, $(-1,2,0,-2,0,0,1)$. (Note that we did the computations in II.) Thus the system of equations for $\text{span}(T)^\perp$ is:

$$
\begin{aligned}
-x_1 \quad\quad +x_3 \quad\quad\quad\quad\quad\quad &= 0\\
-x_1 + 2x_2 \quad - 2x_4 + x_5 \quad\quad\quad &= 0\\
x_2 \quad - 2x_4 \quad + x_6 \quad &= 0\\
-x_1 + 2x_2 \quad - 2x_4 \quad\quad\quad + x_7 &= 0
\end{aligned}
$$

V. Let S and T be the sets from III and IV respectively. Set $V = \operatorname{span}(T)$ and $W = \operatorname{span}(S)$. Find a basis of $V + W$ and $V \cap W$.

Solution: Since $V + W = \operatorname{span}(T \cup S)$ we find a basis of $V + W$ by applying I. Thus we start with a matrix whose columns are the vectors from $T \cup S$:

$$
\left(\begin{array}{cccc|ccccc}
1 & 1 & 3 & 2 & 0 & 1 & 2 & 2 & 1\\
1 & 0 & 2 & -1 & 1 & 1 & 3 & 1 & 2\\
1 & 1 & 3 & 2 & 1 & 1 & 3 & 2 & 2\\
1 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0\\
1 & 3 & 3 & 2 & 1 & -1 & -1 & 0 & 0\\
1 & 2 & 2 & -1 & 1 & -1 & -1 & -1 & 0\\
1 & 3 & 3 & 2 & 0 & 0 & -2 & 2 & 1
\end{array}\right)
$$

Its reduced row-echelon form is

$$
\left(\begin{array}{cccc|ccccc}
1 & 0 & 0 & -7 & 0 & 0 & -2 & 0 & 1\\
0 & 1 & 0 & 0 & 0 & 0 & -2 & 2 & 1\\
0 & 0 & 1 & 3 & 0 & 0 & 2 & -1 & -1\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

The pivot columns of this matrix tell us that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{w}_1, \mathbf{w}_2$ is a basis for $V + W$.

In order to find a basis of $V \cap W$ we may use one of the following methods.

Method 1. Find equations for V and W using IV. We see that the equations for V are

$$
\begin{aligned}
-x_1 \quad\quad +x_3 \quad\quad\quad\quad\quad\quad &= 0\\
-x_1 + 2x_2 \quad - 2x_4 + x_5 \quad\quad\quad &= 0\\
x_2 \quad - 2x_4 \quad + x_6 \quad &= 0\\
-x_1 + 2x_2 \quad - 2x_4 \quad\quad\quad + x_7 &= 0
\end{aligned}
$$

(we found them in IV).

In III we found a basis for $\operatorname{span}(W)^\perp$. Thus W is a solution space to the system

$$
\begin{aligned}
x_4 \quad\quad &= 0\\
2x_1 + 2x_2 - 3x_3 \quad + x_5 \quad &= 0\\
2x_1 + x_2 - 2x_3 \quad\quad + x_6 &= 0
\end{aligned}
$$

Thus $V \cap W$ is the solution space to the combined system

$$
\begin{aligned}
-x_1 \quad\quad +x_3 \quad\quad\quad\quad\quad\quad &= 0\\
-x_1 + 2x_2 \quad - 2x_4 + x_5 \quad\quad\quad &= 0\\
x_2 \quad - 2x_4 \quad + x_6 \quad &= 0\\
-x_1 + 2x_2 \quad - 2x_4 \quad\quad\quad + x_7 &= 0\\
x_4 \quad\quad\quad &= 0\\
2x_1 + 2x_2 - 3x_3 \quad + x_5 \quad\quad &= 0\\
2x_1 + x_2 - 2x_3 \quad\quad + x_6 \quad &= 0
\end{aligned}
$$

To get a basis of $V \cap W$ we just need to find a basis of solutions to this system. The coefficient matrix of this system is

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 & 0 \\ -1 & 2 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 2 & -3 & 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The reduced row-echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Using the method of II we see that a basis of solutions is $(-2, -1, -2, 0, 0, 1, 0)$, $(1, 0, 1, 0, 1, 0, 1)$. This is a basis for $V \cap W$.

Method 2. This method works best if S and T are linearly independent. We could first apply I to find a basis of V and W. But it is more efficient to do everything at once. We start with a matrix whose columns are the vectors from T followed by the vectors from S:

$$\left(\begin{array}{cccc:ccccc} 1 & 1 & 3 & 2 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 2 & -1 & 1 & 1 & 3 & 1 & 2 \\ 1 & 1 & 3 & 2 & 1 & 1 & 3 & 2 & 2 \\ 1 & 1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 2 & 1 & -1 & -1 & 0 & 0 \\ 1 & 2 & 2 & -1 & 1 & -1 & -1 & -1 & 0 \\ 1 & 3 & 3 & 2 & 0 & 0 & -2 & 2 & 1 \end{array}\right)$$

The reduced row-echelon form of this matrix is

$$A = \left(\begin{array}{cccc:ccccc} 1 & 0 & 0 & -7 & 0 & 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & 3 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In the left part we remove all the non-pivot columns (they are combinations of pivot columns and are not needed; it corresponds to the fact that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis of V). The right part is not in the reduced row-echelon form. Its reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This tells us that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ form a basis of W and we can remove the last (non-pivot) column of the right hand side of A. Thus we get a matrix

$$\left(\begin{array}{ccc:cccc} 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The homogeneous system with this matrix as coefficient matrix has the following basis of solutions: $(-2, -2, 2, 1, 0, 1, 0)$, $(0, 2, -1, 0, 3, 0, 1)$, This implies that $-2\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3 = (2, 2, 2, 0, -2, -2, -2)$ and $2\mathbf{v}_2 - \mathbf{v}_3 = (-1, -2, -1, 0, 3, 2, 3)$ form a basis of $V \cap W$.

General Remark. In order to know the pivot columns of a matrix we do not need to get reduced row-echelon form. Any row-echelon form (not necessarily reduced) will work.