## Homework 12, solutions

**Solution to Problem 4.** a) Let $p(n)$ be the number of distinct prime divisors of $n$. Clearly, if $m, n$ are relatively prime, then they do not share any prime divisors. Thus $p(mn) = p(m) + p(n)$. The function $\rho$ is defined by $\rho(n) = 2^{p(n)}$. Thus, for $m, n$ relatively prime we have

$$\rho(mn) = 2^{p(mn)} = 2^{p(m)+p(n)} = 2^{p(n)}2^{p(m)} = \rho(m)\rho(n)$$

which shows that $\rho$ is multiplicative. If $p$ is a prime number, then $\rho(p) = 2 = \rho(p^2)$. This shows that $\rho$ is not completely multiplicative, as $\rho(p^2)$ is not equal to $\rho(p)^2$.

b) We have $f = \rho * \mathbb{1}$. Thus $f$ is multiplicative, beeing a convolution of two multiplicative functions. For any prime $p$ and $k > 0$ we have $\rho(p^k) = 2$. Thus,

$$f(p^k) = \sum_{i=0}^{k} \rho(p^i) = 1 + 2k.$$

It follows that

$$f(p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}) = (1 + 2a_1)(1 + 2a_2) \dots (1 + 2a_m).$$

**Solution to Problem 7.** a) Clerly if $m$ is a product of $k$ prime numbers and $n$ is a product of $l$ prime numbers then $mn$ is a product of $k + l$ prime numbers. Thus

$$\lambda(mn) = (-1)^{k+l} = (-1)^k(-1)^l = \lambda(m)\lambda(n)$$

so $\lambda$ is completely multiplicative.

b) We have $F = \lambda * \mathbb{1}$. Thus $F$ is multiplicative. For a prime power $p^k$ we have $\lambda(p^k) = (-1)^k$. Thus

$$F(p^k) = \sum_{i=0}^{k} \lambda(p^i) = \sum_{i=0}^{k} (-1)^i = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

It follows that $F(p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}) = 1$ if all the exponents $a_i$ are even and $F(p_1^{a_1} p_2^{a_2} \dots p_m^{a_m}) = 0$ otherwise. In other words, $F(n) = 1$ when $n$ is a square and $F(n) = 0$ otherwise.

**Solution to Problem 20.** a) Let $p$ be a prime number. If $p \nmid m$ then $\gcd(p, m) = 1$ so $\phi(pm) = \phi(p)\phi(m) = (p-1)\phi(m)$. If $p|m$ then $m = p^s k$ for some $s > 0$ and $k$ such that $p \nmid k$. Thus $\phi(m) = \phi(p^s)\phi(k) = p^{s-1}(p-1)\phi(k)$ and $\phi(pm) = \phi(p^{s+1})\phi(k) = p^s(p-1)\phi(k)$. It follows that $\phi(pm) = p\phi(m)$ in this case.

We see that $\phi(m)|\phi(pm)$ for any prime number $p$ and any positive integer $m$. Now if $m|n$ then $n = p_1 p_2 \dots p_t m$ for some prime numbers $p_1, \dots, p_t$. Thus,

$$\phi(m)|\phi(p_1 m)|\phi(p_1 p_2 m)| \dots |\phi(p_1 p_2 \dots p_t m) = \phi(n)$$

i.e. $\phi(m)|\phi(n)$.

b) The converse of part a) is false. For example, $\phi(4) = 2$, $\phi(5) = 4$ so $\phi(4)|\phi(5)$ but clearly $4 \nmid 5$.

**Solution to Problem 21.** a) We have seen in the solution to problem 20a) that if $p$ is a prime and $p|n$ then $\phi(pn) = p\phi(n)$.

Let $m = p_1 p_2 \dots p_t$. Since $m|n$, each prime $p_i$ divides $n$. Thus

$$\phi(mn) = \phi(p_1 p_2 \dots p_t n) = p_1 \phi(p_2 \dots p_t n) = p_1 p_2 \phi(p_3 \dots p_t n) = \dots = p_1 p_2 \dots p_t \phi(n) = m\phi(n).$$

**Remark.** We proved a stronger result, namely that if every prime divisor of $m$ divides $n$ then $\phi(mn) = m\phi(n)$.

b) The convesre to part a) is false. Indeed, $\phi(4 \cdot 6) = 8 = 4\phi(6)$ but $4 \nmid 6$.

**Remark.** However, the converse to the stronger result in the remark above is true: if $\phi(mn) = m\phi(n)$ then evry prime divisor of $m$ divides $n$.

**Solution to Problem 28.** Consider the set $S = \{1, 2, \dots, n\}$. For $d|n$, let $S_d$ be the subset of $S$ consisting of those integers $m$ such that $\gcd(m, n) = \frac{n}{d}$. If $m \in S_d$ then $m = \frac{n}{d} a$ for some $a$ such that $1 \le a \le d$ and $\gcd(a, d) = 1$. Conversely, if $1 \le a \le d$ and $\gcd(a, d) = 1$ then $a\frac{n}{d} \in S_d$. It follows that

$$\sum_{m \in S_d} m^k = \sum_{1 \le a \le d, \gcd(a,d)=1} \left(a\frac{n}{d}\right)^k = n^k \frac{\phi_k(d)}{d^k}.$$

Clearly, every element of $S$ belongs to exactly one of the subsets $S_d$. Thus

$$1^k + 2^k + \dots + n^k = \sum_{d|n} \left( \sum_{m \in S_d} m^k \right) = \sum_{d|n} n^k \frac{\phi_k(d)}{d^k} = n^k \sum_{d|n} \frac{\phi_k(d)}{d^k}.$$

**Solution to Problem 40.** The problem asks as to prove that $(\nu * \mathbb{1}(n))^2 = (\nu^3 * \mathbb{1})(n)$ for every $n$. As $\nu$ is multipliacative, so are $\nu^3$, $(\nu * \mathbb{1})^2$, and $\nu^3 * \mathbb{1}$. It suffices then to show that $(\nu * \mathbb{1}(p^k))^2 = (\nu^3 * \mathbb{1})(p^k)$ for every prime power $p^k$. Now

$$(\nu * \mathbb{1})(p^k) = \sum_{i=0}^{k} \nu(p^i) = \sum_{i=0}^{k} (i+1) = \frac{(k+1)(k+2)}{2}$$

and

$$(\nu^3 * \mathbb{1})(p^k) = \sum_{i=0}^{k} \nu^3(p^i) = \sum_{i=0}^{k} (i+1)^3.$$

It suffices then to prove that for every positive integer $n$ we have

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}.$$

This can be done, for example, by induction: if

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

then

$$1^3 + 2^3 + \ldots + n^3 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + n + 1\right) =$$

$$= (n+1)^2 \frac{(n+2)^2}{4} == \frac{(n+1)^2(n+2)^2}{4}.$$

**Solution to Problem 51.** a)

$$\sigma_3(12) = 1^3 + 2^3 + 3^3 + 4^3 + 6^3 + 12^3 = 2044.$$

$$\sigma_4(8) = 1^4 + 2^4 + 4^4 + 8^4 = 4361.$$

b) Let $f_k(n) = n^k$. Then $f_k$ is multiplicative (even completely multiplicative) and $\sigma_k = f_k * \mathbb{1}$. It follows that $\sigma_k$ is multiplicative.

c) Now

$$\sigma_k(p^a) = \sum_{i=0}^{a} p^{ik} = \sum_{i=0}^{a} (p^k)^i = \frac{p^{k(a+1)} - 1}{p^k - 1}.$$

d) Since $\sigma_k$ is multiplicative, part c) yields

$$\sigma_k(p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}) = \frac{p_1^{k(a_1+1)} - 1}{p_1^k - 1} \frac{p_2^{k(a_2+1)} - 1}{p_2^k - 1} \ldots \frac{p_r^{k(a_r+1)} - 1}{p_r^k - 1}.$$