Homework 14, solutions

Solution to Problem 8. Suppose that $e = a/b$ is rational. Then for any $n \geq b$ the number $n!e = n!a/b$ is an integer (since $b|n!$). Pick one such n. Clearly $n!/k!$ is an integer for $k \leq n$. Thus the number

$$\alpha = n!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \ldots - \frac{1}{n!}\right)$$

is an integer. As $e = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}$, the number α is clearly positive, hence it is a positive integer. It follows that $\alpha \geq 1$. On the other hand,

$$\alpha = n! \sum_{k=n+1}^{\infty} \frac{1}{k!}.$$

Note that for $k > n$ we have $\frac{n!}{k!} = \frac{1}{(n+1)(n+2)\ldots k} \leq \left(\frac{1}{n+1}\right)^{k-n}$. Thus

$$\alpha = \sum_{k=n+1}^{\infty} \frac{n!}{k!} \leq \sum_{k=n+1}^{\infty} \left(\frac{1}{n+1}\right)^{k-n} = \sum_{k=1}^{\infty} \left(\frac{1}{n+1}\right)^{k} = \frac{1}{n+1} \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{n}.$$

We see that α is both at least 1 and less than $1/n$, which is a clear contradiction. This shows that our assumption that e is rational was wrong, i.e. e is an irrational number.

Remark. It is known that e is not only irrational, but it is transcendental, which means that $f(e) \neq 0$ for every non-zero polynomial f with integer coefficients. The proof of this is much harder though.

Solution to Problem 9. e) We use Euclidean algorithm:

$$156 = 3 \cdot 49 + 9; \quad 49 = 5 \cdot 9 + 4, \quad 9 = 2 \cdot 4 + 1, \quad 4 = 4 \cdot 1 + 0.$$

Thus $\frac{156}{49} = [3, 5, 2, 4]$.

f) We use Euclidean algorithm:

$$64 = 0 \cdot 391 + 64, \quad 391 = 6 \cdot 64 + 7, \quad 64 = 9 \cdot 7 + 1, \quad 7 = 7 \cdot 1 + 0.$$

Thus $\frac{64}{391} = [0, 6, 9, 7]$.

Solution to Problem 11. e)

$$[0,2,4,6,8] = 0 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8}}}} = \frac{204}{457}.$$

f)

$$[9,9,9,9] = 9 + \cfrac{1}{9 + \cfrac{1}{9 + \cfrac{1}{9}}} = \frac{6805}{747}.$$

Solution to Problem 17. We will use the recursion

$$p_n = k_n p_{n-1} + p_{n-2}, p_{-1} = 1, p_0 = k_0, \quad \text{and} \quad q_n = k_n q_{n-1} + q_{n-2}, q_{-1} = 0, q_0 = 1.$$

Then $p_0/q_0, p_1/q_1, \ldots, p_n/q_n$ are the convergents for the continued fraction $[k_0, k_1, \ldots, k_n]$.

d) We have $k_0 = 0,\ k_1 = 1,\ k_2 = 1,\ k_3 = 1,\ k_4 = 1,\ k_5 = 1,\ k_6 = 1, k_7 = 4$. Thus

$$p_0 = 0, p_1 = 1, p_2 = 1, p_3 = 2, p_4 = 3, p_5 = 5, p_6 = 8, p_7 = 37,$$

and

$$q_0 = 1, q_1 = 1, q_2 = 2, q_3 = 3, q_4 = 5, q_5 = 8, q_6 = 13, q_7 = 60.$$

The convergents are $0, 1, 1/2, 2/3, 3/5, 5/8, 8/13, 37/60$. We have

$$0 < \frac{1}{2} < \frac{3}{5} < \frac{8}{13} < \frac{37}{60} < \frac{5}{8} < \frac{2}{3} < 1.$$

e) We have $k_0 = 3,\ k_1 = 5,\ k_2 = 2,\ k_3 = 4$. Thus

$$p_0 = 3, p_1 = 16, p_2 = 35, p_3 = 156,$$

and

$$q_0 = 1, q_1 = 5, q_2 = 11, q_3 = 49.$$

The convergents are $3, 16/5, 35/11, 156/49$. We have

$$3 < \frac{35}{11} < \frac{156}{49} < \frac{16}{5}.$$

f) We have $k_0 = 0$, $k_1 = 6$, $k_2 = 9$, $k_3 = 7$. Thus

$$p_0 = 0, p_1 = 1, p_2 = 9, p_3 = 64,$$

and

$$q_0 = 1, q_1 = 6, q_2 = 55, q_3 = 391.$$

The convergents are $0, 1/6, 9/55, 64/391$. We have

$$0 < \frac{9}{55} < \frac{64}{391} < \frac{1}{6}.$$

Solution to Problem 18. Clearly $p_0/p_{-1} = a_0$ and $q_1/q_0 = a_1$. Suppose for some $i \geq 0$ we have

$$[a_i, a_{i-1}, \ldots, a_0] = \frac{p_i}{p_{i-1}}.$$

Then

$$[a_{i+1}, a_i, a_{i-1}, \ldots, a_0] = a_{i+1} + \frac{1}{[a_i, a_{i-1}, \ldots, a_0]} = a_{i+1} + \frac{p_{i-1}}{p_i} = \frac{a_{i+1}p_i + p_{i-1}}{p_i} = \frac{p_{i+1}}{p_i}.$$

Similarly, if for some $i \geq 1$ we have

$$[a_i, a_{i-1}, \ldots, a_1] = \frac{q_i}{q_{i-1}}$$

then

$$[a_{i+1}, a_i, a_{i-1}, \ldots, a_1] = a_{i+1} + \frac{1}{[a_i, a_{i-1}, \ldots, a_1]} = a_{i+1} + \frac{q_{i-1}}{q_i} = \frac{a_{i+1}q_i + q_{i-1}}{q_i} = \frac{q_{i+1}}{q_i}.$$

Thus the result follows by induction.

Solution to Problem 33. e) Let $x = [2, 3, 4, 2, 3, 4, \ldots]$. Then

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{x}}} = \frac{30x + 7}{13x + 3}.$$

Thus $x(13x+3) = 30x+7$, i.e. $13x^2 - 27x - 7 = 0$. The solutions to this equations are $(27 \pm \sqrt{1093})/26$. Since $x > 2$, we have $x = (27+\sqrt{1093})/26$.

f) Let $x = [3,4,3,4,\ldots]$. Thus $x = 3 + \frac{1}{4+\frac{1}{x}} = \frac{13x+3}{4x+1}$. Thus $x(4x+1) = 13x+3$, i.e. $4x^2 - 12x - 3 = 0$. The roots of this equation are $(3 \pm 2\sqrt{3})/2$, so $x = (3+2\sqrt{3})/2$, as $x > 3$.

Now

$$[1,2,\overline{3,4}] = [1,2,x] = 1 + \cfrac{1}{2 + \cfrac{1}{x}} = 1 + \frac{x}{2x+1} = 1 + \frac{(3+2\sqrt{3})/2}{4+2\sqrt{3}} = \frac{4+\sqrt{3}}{4}.$$