Homework 15

due on Friday, April 27

Study carefully Chapter 7 of the book. Solve problems 19, 36, 38, 39, 40 in Chapter 7.

Remark. Lemma 7.22 in the bookis false as stated. It is true if $b_1 = b_2$. This remark only applies to the older edition of the book, it was corrected in the never edition.

Also solve the following problems.

Problem 1. Let $d > 1$ be an integer which is not a square (so $\sqrt{d}$ is irrational). Suppose that positive integers m, n satisfy $m^2 - n^2 d = \pm 1$. Prove that m/n is a convergent of $\sqrt{d}$.

Hint Show that $|\sqrt{d} - m/n| < 1/2n^2$.

Problem 2. Let x be an irrational number, let the simple continued fraction expression for x be $x = [k_0, k_1, \ldots]$, and let p_i/q_i, $i = 0, 1 \ldots$ be the convergents of x.

a) Prove that if $n > 0$ then

$$\frac{1}{(k_{n+1}+2)q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{k_n q_n^2}.$$

b) Prove that there is $c > 0$ such that $|x - \frac{a}{b}| > \frac{1}{cb^2}$ for every fraction a/b if and only if the sequence $k_0, k_1, \ldots$ is bounded. Numbers x with this property are called **poorly approximable**.