Homework 1, solutions

Problem 1. Suppose that $a_1 = 2$ and $a_{n+1} = 3a_n + 2$. Prove that $a_n = 3^n - 1$ for every natural number n.

Solution: We prove that $a_n = 3^n - 1$ by induction on n. For $n = 1$ we have $a_1 = 2 = 3^1 - 1$, so the result holds for $n = 1$. Assume that $n \geq 1$ and $a_k = 3^k - 1$ for $k = 1, 2, \ldots, n$. In particular, $a_n = 3^n - 1$. Therefore $a_{n+1} = 3a_n + 2 = 3(3^n - 1) + 2 = 3^{n+1} - 1$. Thus the formula holds for $n+1$. By the method of induction, the formula holds for every integer $n \geq 1$.

Remark. In the inductive step for this problem we only needed to know that the formula holds for n in order to conclude that it holds for $n + 1$.

Problem 2. Prove by induction that every natural number n is a sum of disctinct powers of 2 (e.g. $1 = 2^0$; $2 = 2^1$, $3 = 2^0 + 2^1$, etc.). Hint: In the inductive step consider the case when your number is even and the case when it is odd.

Extra credit: prove that such expression is unique. Hint: Observe that $1 + 2 + 4 + .. + 2^n < 2^{n+1}$

Solution: We prove this by induction on n. The problem provides a verification of the result for $n = 1, 2, 3$. Suppose that $n \geq 1$ and the result is true for $1, 2, \ldots, n$. We want to justify that the result is true for $n + 1$. Note that $n + 1$ is either even or odd.

case 1: $n + 1 = 2k$ is even. Then $1 \leq k \leq n$, so we know that the result holds for k. In other words, $k = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_s}$ for some integer $s \geq 1$ and integers $0 \leq m_1 < m_2 < \ldots < m_s$. But then

$$n + 1 = 2k = 2(2^{m_1} + 2^{m_2} + \ldots 2^{m_s}) = 2^{m_1+1} + 2^{m_2+1} + \ldots 2^{m_s+1}$$

is a sum of distinct powers of 2, so the result holds for $n + 1$.

case 2: $n + 1 = 2k + 1$ is even. Then again $1 \leq k \leq n$, so we know that the result holds for k, as in the first case. In other words, $k = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_s}$ for some integer $s \geq 1$ and integers $0 \leq m_1 < m_2 < \ldots < m_s$. But then

$$n + 1 = 1 + 2k = 2^0 + 2(2^{m_1} + 2^{m_2} + \ldots + 2^{m_s}) = 2^0 + 2^{m_1+1} + 2^{m_2+1} + \ldots 2^{m_s+1}.$$

Since $0 < m_1 + 1 < m_2 + 1 < \ldots < m_s + 1$, $n + 1$ is a sum of distinct powers of 2, so the result holds for $n + 1$.

We proved the result for $n+1$ in both cases, hence, by the method of induction, the result is true for all integers $n \geq 1$.

Second proof. Here we giva a differnt inductive proof. As before, suppose that $n \geq 1$ and the result is true for $1, 2, \ldots, n$. We want to justify that the result is true for $n+1$. There is an integer $k \geq 1$ such that $2^k \leq n+1 < 2^{k+1}$. If $n+1 = 2^k$, the result clearly holds for $n+1$. Otherwise, we have $0 < n+1-2^k < 2^k < n+1$. It follows that the result holds for $n+1-2^k$, i.e. $n+1-2^k = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_s}$ for some integer $s \geq 1$ and integers $0 \leq m_1 < m_2 < \ldots < m_s$. Clearly $m_s < k$ (since $n+1-2^k < 2^k$), so $n+1 = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_s} + 2^k$ and the result holds for $n+1$.

Suppose now that two different sums of distinct powers of 2 add to the same number. Thus, we have

$$2^{m_1} + 2^{m_2} + \ldots + 2^{m_s} = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_t}$$

for some integers $s, t \geq 1$ and integers $0 \leq m_1 < m_2 < \ldots < m_s$, $0 \leq n_1 < n_2 < \ldots < n_t$. We may assume that $m_s \neq n_t$ (if $m_s = n_t$, we can cancel 2^{m_s} from both sides and still have two different sums of distinct powers of 2 adding to the same number). Without any loss of generality, we may assume that $m_s < n_t$. It follows that $2^{n_1} + 2^{n_2} + \ldots + 2^{n_t} \geq 2^{n_t}$ and

$$2^{m_1} + 2^{m_2} + \ldots + 2^{m_s} \leq 2^0 + 2^1 + 2^2 + \ldots + 2^{m_s} = 2^{m_s+1} - 1 < 2^{m_s+1} \leq 2^{n_t}$$

which contradicts the equality $2^{m_1} + 2^{m_2} + \ldots 2^{m_s} = 2^{n_1} + 2^{n_2} + \ldots 2^{n_t}$.

Problem 3. We defined in class $v(n)$ to be the number of positive divisors of n. Charcterize positive integers n such that $v(n) = 3$.

Solution: Since $v(n) = 3$, we must have $n > 1$ and n is not a prime. It follows that $n = pm$ for some prime number p and integer $m \geq 2$. Note that $1, p, m, pm$ are divisors of n, hence two of them must be equal. The only way this can happen is when $p = m$, so $n = p^2$ is a suqare of a prime. Conversely, if $n = p^2$ for some prime p then $1, p, p^2$ are the only divisors of n, so $v(n) = 3$.