Homework

due on Wednesday, October 13

Read carefully Chapter 2 of Hartshorne's book. Solve problems 6.1, 6.3 a and c, 7.6, 7.8 in Chapter 2. Work on other problems to sections 6 and 7.

Also solve the following problems.

Problem 1. In class we proved the following theorem.

Theorem. Let ABC be a triangle and let P be a point whose orthogonal projections on the sides of the triangle are K, L, M. Then the points K, L, M are collinear if and only if P is on the circumscribed circle of the triangle ABC. The line through K, L, M is then called **the Simson line of** P.

Using this result solve the following problem.

Let A, B, C be three collinear points and let P be a point outside the line through A, B, C. Prove that the circumcenters of the triangles PAB, PAC, PBC and the point P lie on a circle. Hint: Note that if two circles intesect at two points X, Y then the line joining the centers of the circles is the perpendicular bisector of XY. Consider the triangle with vertices at the circumcenters. What are the projections of P on the sides of this triangle?

Problem 2. Let ABC be a triangle and let a line l intersect the lines AB, BC, AC at C_1, A_1, B_1 respectively. Let P be the corresponding Miquel point. Prove that the orthogonal projections of P on the lines AB, AC, BC and l are collinear.

Problem 3. The altitudes of a triangle ABC intersect at a point H. Let O_A be the circumcenter of the triangle BCH. Similarly define O_B and O_C. Prove that the segments AO_A, BO_B, CO_C share a common midpoint. What is this point? Conclude that the triangles ABC and $O_AO_BO_C$ are congruent.

Hint: 1) What is the orthocenter of BCH?

2) What can you say about the nine-point circles of the triangles ABC, ABH, BCH, ACH?

b) Prove that H is the circumcenter of $O_AO_BO_C$ and that the circumcenter of ABC is the orthocenter of $O_AO_BO_C$.

c) Prove that the Euler lines of the triangles ABC, ABH, BCH, ACH intersect at one point.

Problem 4. The construction of Problem 5.11 in section 5 of Chapter 1 in the book works only if the point F exists. Here is a construction which always works (assuming that A, B are either both inside or both outside of Γ).

If the perpendicular bisector of AB passes through the center of the given circle Γ then the construction is easy (explain what to do in this case). Suppose then that the perpendicular bisector of AB does not contain the center of Γ. Take an arbitrary point E on Γ which is not on the line AB. Construct the circumscribed circle of the triangle ABE. This circle intersects Γ at points E and D. Let M be the point of intersection of the lines AB and DE (why do these lines intersect? what should you do if $D = E$?). Construct a point C on Γ such that the line MC is tangent to Γ. Then C is the required point (note that usually there are 2 solutions).

Prove that this construction is correct. Hint: Prove that the line MC is tangent to the circumcircle of the triangle ABC (Propositions 36-37 of Elements, Book III should be helpful).