

## Homework

due on Monday, October 27

Read carefully Chapter 2 of Hartshorne's book. Solve problems 7.1, 7.10, 8.2, 10.6, 10.9, 11.3, 12.1. Also solve the following problems.

**Problem 1.** Consider an incidence geometry satisfying the betweenness axioms  $B1 - B4$ . Let  $A, B, C$  be points on a line  $l$  such that  $A * B * C$  (i.e.  $B$  is between  $A$  and  $C$ ). Let  $t_A, t_B, t_C$  be lines through  $A, B, C$  respectively such that  $t_A$  and  $t_B$  are parallel and distinct and  $t_C$  and  $t_B$  are parallel and distinct (note: this does not imply that  $t_A$  and  $t_C$  are parallel). Let  $m$  be a line intersecting lines  $t_A, t_B, t_C$  at points  $X, Y, Z$  respectively. Prove that  $X * Y * Z$ . Hint: Use sides of the line  $t_B$ .

**Problem 2.** A subset  $S$  of a plane satisfying incidence and betweenness axioms is called **convex** if for any two points  $X, Y$  in  $S$  the whole segment  $\overline{XY}$  is contained in  $S$ .

- a) Prove that intersection of convex sets is convex.
- b) Prove that any ray and a side of any line are convex sets.
- c) Prove that a segment, an interior of an angle, an interior of a triangle are convex (use a) and b)).
- d) In any Hilbert plane, prove that an interior of a circle is convex. Hint: Prove first that if  $ABC$  is a triangle and  $X$  is between  $A$  and  $B$  then either  $\overline{AX} < \overline{AB}$  or  $\overline{AX} < \overline{AC}$ . You may use propositions 2-27 from book 1 of Euclid.

**Problem 3.** Recall that an incidence geometry is called a **projective plane** if it satisfies the following 2 axioms:

- (PP1) any two lines intersect.
- (PP2) there exist 4 points, no three of which are on one line.

Let  $\Pi$  be a projective plane.

- a) Prove that every line in  $\Pi$  has at least 3 points.
- b) Let  $l, m$  be two lines in  $\Pi$ . Prove that there is a point  $A$  not belonging to either

$l$  or  $m$ . Use lines through  $A$  to construct a bijection between points on  $l$  and points on  $m$ .

c) Suppose that  $\Pi$  is finite. Then, by a) and b), there is  $n \geq 2$  such that every line has exactly  $n + 1$  points. Prove that  $\Pi$  has  $n^2 + n + 1$  points and  $n^2 + n + 1$  lines.

Choose a line  $l$  in  $\Pi$ .

d) Let  $\Pi_1$  be the set of all points in  $\Pi$  which are not on  $l$ . Call a subset of  $\Pi_1$  a line if it is equal to an intersection of a line in  $\Pi$  with  $\Pi_1$ . Prove that  $\Pi_1$  is an affine plane. (This allows to prove c) by using results about finite affine planes)

e) Do problem 6.7 a) (which shows that conversely, any affine plane can be extended to a projective plane by adding "the line at infinity").

f) Let  $\Pi$  be a projective plane. Consider the set  $\Pi^*$  of all lines in  $\Pi$ . A line in  $\Pi^*$  is defined as a set of all lines in  $\Pi$  passing through a given point of  $\Pi$  (so lines in  $\Pi^*$  are naturally identified with points in  $\Pi$ ). Prove that  $\Pi^*$  is also a projective plane (called the dual plane to  $\Pi$ ).