Homework

due on Monday, October 27

Read carefully Chapter 2 of Hartshorne's book. Solve problems 7.1, 7.10, 8.2, 10.6, 10.9, 11.3, 11.6, 12.1. Also solve the following problems.

Problem 1. Consider an incidence geometry satisfying the betweenness axioms $B1 - B4$. Let A, B, C be points on a line l such that $A * B * C$ (i.e. B is between A and C). Let t_A, t_B, t_C be lines through A, B, C respectively such that t_A and t_B are parallel and distinct and t_C and t_B are parallel and distinct (note: this does not imply that t_A and t_C are parallel). Let m be a line intesecting lines t_A, t_B, t_C at points X, Y, Z respectively. Prove that $X * Y * Z$. Hint: Use sides of the line t_B.

Problem 2. A subset S of a plane satisfying incidence and betweenness axioms is called **convex** if for any two points X, Y in S the whole segment $\overline{XY}$ is contained in S.

a) Prove that intersection of convex sets is convex.

b) Prove that any ray and a side of any line are convex sets.

c) Prove that a segment, an interior of an angle, an interior of a triangle are convex (use a) and b)).

d) In any Hilbert plane, prove that an interior of a circle is convex. Hint: Prove first that if ABC is a triangle and X is between A and B then either $\overline{AX} < \overline{AB}$ or $\overline{AX} < \overline{AC}$. You may use propositions 2-27 from book 1 of Euclid.

Problem 3. Recall that an incidence geometry is called a **projective plane** if it satifies the following 2 axioms:

(PP1) any two lines intersect.

(PP2) there exist 4 points, no three of which are on one line.

Let Π be a projective plane.

a) Prove that every line in Π has at least 3 points.

b) Let l, m be two lines in Π. Prove that there is a point A not belonging to either

l or m. Use lines through A to construct a bijection between points on l and points on m.

c) Suppose that Π is finite. Then, by a) and b), there is $n \geq 2$ such that every line has exactly $n+1$ points. Prove that Π has n^2+n+1 points and n^2+n+1 lines.

Choose a line l in Π.

d) Let Π_1 be the set of all points in Π which are not on l. Call a subset of Π_1 a line if it is equal to an intersection of a line in Π with Π_1. Prove that Π_1 is an affine plane. (This allows to prove c) by using results about finite affine planes)

e) Do problem 6.7 a) (which shows that conversly, any affine plane can be extended to a projective plane by adding "the line at infinity").

f) Let Π be a projective plane. Consider the set Π^* of all lines in Π. A line in Π^* is defined as a set of all lines in Π passing through a given point of Π (so lines in Π^* are naturally identified with points in Π). Prove that Π^* is also a projective plane (called the dual plane to Π).