Homework

due on Wednesday, October 16

Read carefully Chapter 1 of Hartshorne’s book. Read Books 1-4 of the ”Elements”. Study the links on the course web-page. Solve the following problems in Chapter 1:

Problem 2.24. Hint: Let $O_1, O_2$ be the centers of the circles, $r_1, r_2$ corresponding radii with $r_2 \geq r_1$. Prove that solution exists iff $O_1 O_2 \geq r_2 - r_1$. Explain why it suffices to construct a right angled triangle $O_1 O_2 D$ with hypotenuse $O_1 O_2$ and $DO_2 = r_2 - r_1$. If $O_1 O_2 \geq r_1 + r_2$ there is also a second solution. Explain how to construct it.

Problem 3.8. Hint: Consider the circle through $A, D, B$.

Problems 3.10, 3.11.

Problem 4.7.

Problem 5.5.

Your solutions should only use the results from books 1-4 of the Elements and Chapter 1 of the book, and you should carefully explain what propositions you are using. Work on several other problems to section 5.

Also solve the following problems.

Problem 1. In class we proved the following theorem.

**Theorem.** Let $ABC$ be a triangle and let $P$ be a point whose orthogonal projections on the sides of the triangle are $K, L, M$. Then the points $K, L, M$ are collinear if and only if $P$ is on the circumscribed circle of the triangle $ABC$. The line through $K, L, M$ is then called the **Simson line of $P$**.

Using this result solve the following problem.

Let $A, B, C$ be three collinear points and let $P$ be a point outside the line through $A, B, C$. Prove that the circumcenters of the triangles $PAB, PAC, PBC$ and the point $P$ lie on a circle. Hint: Note that if two circles intesect at two points $X, Y$ then the line joining the centers of the circles is the perpendicular bisector of $XY$. Consider the triangle with vertices at the circumcenters. What are the projections of $P$ on the sides of this triangle?
**Problem 2.** Let $ABC$ be a triangle and let a line $l$ intersect the lines $AB$, $BC$, $AC$ at $C_1$, $A_1$, $B_1$ respectively. Let $P$ be the corresponding Miquel point. Prove that the orthogonal projections of $P$ on the lines $AB$, $AC$, $BC$ and $l$ are collinear.

**Problem 3.** The altitudes of a triangle $ABC$ intersect at a point $H$. Let $O_A$ be the circumcenter of the triangle $BCH$. Similarly define $O_B$ and $O_C$. Prove that the segments $AO_A$, $BO_B$, $CO_C$ share a common midpoint. What is this point? Conclude that the triangles $ABC$ and $O_AO_BO_C$ are congruent.

Hint: 1) What is the orthocenter of $BCH$?

2) What can you say about the nine-point circles of the triangles $ABC$, $ABH$, $BCH$, $ACH$?

b) Prove that $H$ is the circumcenter of $O_AO_BO_C$ and that the circumcenter of $ABC$ is the orthocenter of $O_AO_BO_C$.

c) Prove that the Euler lines of the triangles $ABC$, $ABH$, $BCH$, $ACH$ intersect at one point.